DOCUMENT RESUME

ED 175 644 SE 027 885

TITLE CSMP Mathematics for the Intermediate Grades Part III, Teacher's Guide. The Languages of Strings and Arrows. Geometry and Measurement. Probability and Statistics. Experimental Version.

INSTITUTION Central Midwestern Regional Educational Lab., St. Ann, Mo.

SPONS AGENCY National Inst. of Education (DHEW), Washington, D.C.

PUB DATE 78

NOTE 289p.; For related documents, see SE 027 876-893; Contains colored charts and activities which may not reproduce well; Not available in hard copy due to copyright restrictions

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.

DESCRIPTORS *Curriculum Guides; *Elementary School Mathematics; Games; Geometry; *Instructional Materials; Intermediate Grades; Mathematical Logic; *Mathematics Curriculum; Mathematics Instruction; Measurement; Probability; Rational Numbers; Statistics; Textbooks; Workbooks

IDENTIFIERS *Comprehensive School Mathematics Program

ABSTRACT

This guide represents the final experimental version of a pilot project which was conducted in the United States between 1973 and 1976. The ideas and the manner of presentation are based on the works of Georges and Frederique Papy. They are recognized for having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. This program follows the "spiral approach." An explanation is provided of the languages of strings and arrows or classification and relations. This guide includes functions as well. The string game is presented as a means of using sets to stimulate logical thinking. Geometry, measurement, probability, and statistics comprise the latter half of the guide. (Author/SA)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

Comprehensive School Mathematics Program
CEMREL, Inc.
Experimental Version
5-25401

CSMP Mathematics for the Intermediate Grades Part III

Teacher's Guide

The Languages of Strings and Arrows
Geometry and Measurement
Probability and Statistics

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY
Clare Heideman
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

ED175644

SE 027 885

CEMREL, Inc.
Comprehensive School Mathematics Program
3120 59th. Street, St. Louis, Mo. 63139

The Comprehensive School Mathematics Program materials included herein are in the process of development. As a part of our continuing effort to evaluate and improve them, we ask that you comment in detail on the materials and on the way in which you used them.

Prepared by CEMREL, Inc., a private nonprofit corporation supported in part as an educational laboratory by funds from the National Institute of Education, Department of Health, Education and Welfare. The opinions expressed in this publication do not necessarily reflect the position or policy of the National Institute of Education, and no official endorsement should be inferred.

Copyright on these materials is claimed only during the period of development, test, and evaluation, unless additional authorization is granted by the National Institute of Education to claim copyright on the final materials. For information on the status of the copyright claim, contact either the copyright proprietor or the National Institute of Education.

Manufactured in the United States of America.

Copyright © 1978 CEMREL, Inc.

THE LANGUAGES OF STRINGS AND ARROWS

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part III

The Languages of Strings and Arrows

CEMREL, Inc.

TABLE OF CONTENTS

SUGGESTED SCHEDULE FOR L - LESSONS vi

SUGGESTED CHRONOLOGICAL ORDER OF L - LESSONS . . . ix

INTRODUCTION xi

Classification: The Language of Strings . . . xi

Relations: The Language of Arrows xvii

CHAPTER 1: RELATIONS, FUNCTIONS, AND COMPOSITION . . . 1

1.1 Composition of Relations 5

Worksheets L01, L02, L03 20

1.2 Addition and Multiplication with Ten Friends . . . 21

Worksheets L04, L05 31

1.3 Powers of ⊗2 32

Worksheets L06, L07 42

1.4 Arrow Messages 43

Worksheets L08, L09, L10, L11 51

1.5 The Kite 52

Worksheets L12, L13 59

1.6 Addition and Multiplication with Eighty Friends . . . 60

Worksheets L14, L15 71

1.7 The Kite with Four Tails #1 72

1.8 The Kite with Four Tails #2 78

Worksheets L16, L17 87

CHAPTER 2: STRING GAMES AND LOGICAL THINKING . . . 89

2.1 String Game Analysis #1 90

Worksheets L18, L19, L20, L21 94

2.2 String Game Analysis #2 95

2.3 String Game Analysis #3 105

Worksheets L22, L23, L24, L25 110

2.4 String Game Analysis #4 111

Worksheets L26, L27, L28, L29, L30 115

APPENDIX: The String Game 117
Equipment 117
Preparation for the Game 120
Object of the Game 121
Rules of the Game 121
Version A: The String Game with A-Blocks 122
Version B: The String Game with A-Blocks, using "NOT" Cards 125
Version C: The String Game with Numbers: Multiples and Order 127
Version D: The String Game with Numbers: Multiples, Divisors, and Order 129
Version E: The String Game with Special Scoring Rules . . 132

SUGGESTED SCHEDULE FOR L - LESSONS

WEEK	Languages of Strings and Arrows	Materials
1	**1L** L1.1 Composition of Relations p. 5	Teacher: Colored chalk; IG-III Language Poster #1 Student: Colored pencils; Worksheets L01, L02, L03
2	**2L** L2.1 String Game Analysis #1 p. 90	T: Felt-tipped pen; The String Game; 2 Demonstration Analysis Sheets S: Pencil; String Game Analysis Pad; Worksheets L18, L19, L20, L21
3	**3L** L1.2 Addition and Multiplication with Ten Friends p. 21	T: None S: Worksheets L04, L05
4	**4L** L1.3 Powers of ⊗2 p. 32	T: Colored chalk S: Colored pencils; Worksheets L06, L07
5	**5L** L2.2 String Game Analysis #2 p. 95	T: Colored chalk; The String Game S: String Game Analysis Pad
6	**6L** L1.4 Arrow Messages p. 43	T: Colored chalk S: Colored pencils; Worksheets L08, L09, L10, L11
7	**7L** L1.5 The Kite p. 52	T: Colored chalk S: Colored pencils; Worksheets L12, L13
8	ADJUSTMENT DAY	
9	**8L** L2.3 String Game Analysis #3 p. 105	T: Colored chalk; The String Game S: String Game Analysis Pad; Worksheets L22, L23, L24, L25

SUGGESTED SCHEDULE FOR L - LESSONS

WEEK	Languages of Strings and Arrows	Materials
10	**9L** L1.6 Addition and Multiplication with Eighty Friends p. 60	T: None S: Worksheets L14, L15
11	ADJUSTMENT DAY	
12	**10L** L2.4 String Game Analysis #4 p. 111	T: Felt-tipped pen; colored chalk; two Demonstration Analysis Sheets; The String Game S: String Game Analysis Pad; Wkshts. L26, L27, L28, L29, L30
13	**11L** L1.7 The Kite with Four Tails #1 p. 72	T: Colored chalk S: Paper; colored pencils
14	**12L** L1.8 The Kite with Four Tails #2 p. 78	T: Colored chalk; IG-III Language Poster #2 S: Paper; colored pencils; Worksheets L16, L17
15	**1P** P1.1 Shunda's Newsstand #1 p. 2	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"
16	**2P** P1.2 Shunda's Newsstand #2 p. 7	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"
17	**3P** P1.3 Shunda's Newsstand #3 p. 12	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"
18	**4P** P1.4 Shunda's Newsstand #4 p. 18	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"

SUGGESTED CHRONOLOGICAL ORDER OF L - LESSONS

Lesson One	L1.1 - Composition of Relations . . .	5
Lesson Two	L2.1 - String Game Analysis #1 . . .	90
Lesson Three	L1.2 - Addition and Multiplication with Ten Friends	21
Lesson Four	L1.3 - Powers of ⊗2	32
Lesson Five	L2.2 - String Game Analysis #2 . . .	95
Lesson Six	L1.4 - Arrow Messages	43
Lesson Seven	L1.5 - The Kite	52
Lesson Eight	L2.3 - String Game Analysis #3 . . .	105
Lesson Nine	L1.6 - Addition and Multiplication with Eighty Friends	60
Lesson Ten	L2.4 - String Game Analysis #4 . . .	111
Lesson Eleven	L1.7 - The Kite with Four Tails #1 . . .	72
Lesson Twelve	L1.8 - The Kite with Four Tails #2 . . .	78

THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings represents, at a nonverbal level, the language of sets, that is, the grouping together of objects whether the objects are numbers or not. The language of arrows, on the other hand, is the nonverbal language of relations among objects of the same or different sets. Each of these languages permeates the different content strands. They are sometimes used separately, sometimes together to present an unending variety of numerical and nonnumerical situations to the student.

Classification: The Language of Strings

Classifying things is one of the most important and all-pervasive of human intellectual activities. It is one of the fundamental tools we use to get a handle on things--to keep them straight in our minds, and to avoid being overwhelmed by a sea of chaos. We classify all the time in our everyday life--often unconsciously or, at least, without much deliberate thought: we classify cars by brand (Ford, Chevrolet, etc.); we classify people by citizenship (American, English, German, Australian); we classify the letters of the alphabet into vowels and consonants, and so on. On a more systematic level, much of science is the search for apt and useful systems of classification: the astronomer sorts out the heavenly bodies into stars, planets, moons, comets; the zoologist sorts out animals into reptiles, birds, mammals, insects, protozoa, etc.; the botanist does the same with plants; the linguist sorts out languages into Germanic, Romance, Slavic, etc. Last but not least, the mathematician is always classifying--he sorts out numbers into even and odd; he categorizes geometric figures (by the number of sides) into triangles, quadrangles, pentagons, etc.; he distinguishes between rational and irrational numbers, and so on.

It is true that our penchant for classification is often misused and abused. Educators, for example, sometimes classify children into "bright" and "slow" when it is by no means clear that this kind of sorting out does not oversimplify a very complex situation and does <u>not</u> serve to bring useful insights. We are all aware of the mischief that simpleminded or even vicious stereotyping of

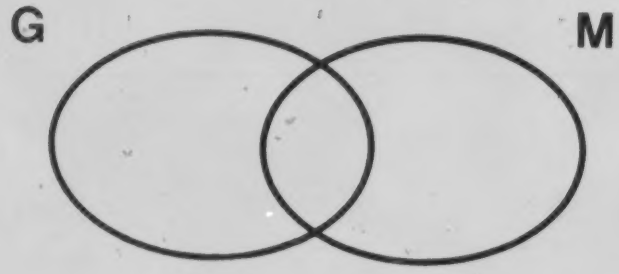
people, ideas, or things can work. However, not only the legitimate uses, but even the abuses of identification schemes point to the importance of teaching children to become proficient at handling the ideas behind classifications. Indeed, if the elementary theory of sets has any place at all in the elementary mathematics curriculum, it is precisely because that theory helps us to understand the basic ideas of classification. It is to this purpose that the language of strings is utilized in the CSMP curriculum.

Classifying means just what it says--putting things into classes, or as the mathematician says: sets. In its simplest context a given thing either belongs to a given set (i.e., has a given property) or it does not. For example, since Abraham Lincoln was a President of the United States, he belongs to the set of American Presidents, while Spiro Agnew does not belong to that set. Similarly, 6 belongs to the set of even numbers and 5 does not belong to the set; or, in other words, 6 does not belong to the set of odd numbers and 5 does belong to the set.

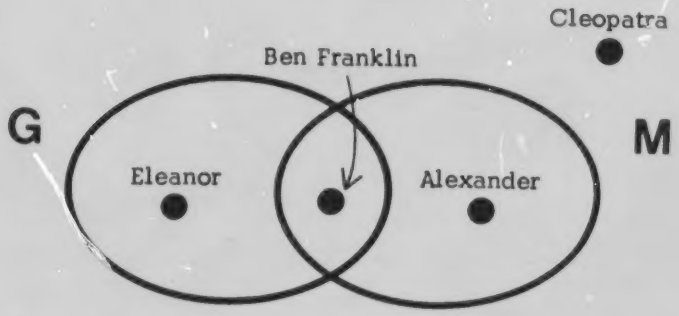
The basic idea is incredibly simple: Given a set S and any thing x, either x belongs to S or x does not belong to S; either x is in S or x is not in S. It is the most basic act of sorting--"in" or "out". We can represent this elementary situation in a neat way by using the so-called "string pictures", which illustrate the idea of "in" or "out" in a very dramatic way. We represent the objects (the things) to be sorted by dots, and we represent the sets into which they are to be sorted by drawing "strings". Consider the Lincoln-Agnew situation referred to above. The string labeled "P" represents the set of Presidents. The dots represent various people. Lincoln is in P and Agnew is out.

Note: A dot by itself can stand for any thing, e.g., Napoleon, the Eiffel Tower, the number 5, the city of Chicago, and so forth. But, once the dot goes inside the P-string it no longer can stand for just anything. It cannot represent the Eiffel Tower or Louis Armstrong; it must be for a President of the United States. Of course that still leaves (as of 1978) 38 choices (from George Washington to Jimmy Carter). Similarly, if a dot is outside the string it could be for the Eiffel Tower, or the Soviet Union, or the planet Mars, or even the number 5, but it cannot be for Thomas Jefferson.

The situation becomes interesting when we are sorting with respect to more than one property, i.e., when we are putting things into several "cubbyholes" (sets); because then we can study the relations between the sets and that is the beginning of the (very) elementary theory of sets. If G is the set of people who wear glasses and M is the set of males, we can draw two (colored) strings.



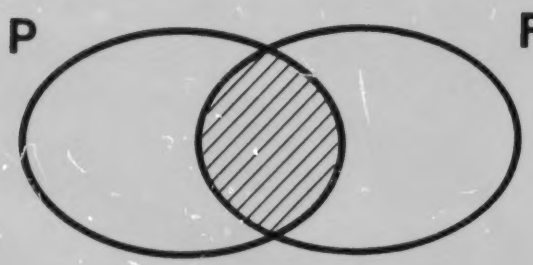
Now the classification scheme has to be concerned with two in/out decisions simultaneously. Consider Cleopatra--where does she go? She's not male and she did not wear glasses (glasses were not invented then!) so she goes outside both strings.



On the other extreme, Ben Franklin was a male who wore glasses, so he has to go inside both the M-string and the G-string. Alexander the Great, by the same argument as the one for Cleopatra, was a man who did not wear glasses so he goes inside the M-string and outside the G-string. Eleanor Roosevelt did wear glasses and was not a male, so she goes inside the G-string and outside the M-string. Note that every object x in the world has its place in the string picture above; there are four possibilities:

1. x is in G and x is in M;
2. x is not in G and x is in M;
3. x is in G and x is not in M;
4. x is not in G and x is not in M.

Of course, there are situations where it is impossible for anything to go into one of the "cubbyholes". For example, if P is the set of all U. S. Presidents and F is the set of all females, then we have (as of 1978) no x such that x is in F and x is in P. We can indicate this state of affairs by hatching the appropriate region--to suggest that no dots should be placed in the region in question.



Thus the language of strings and dots gives children a precise (and nonverbal) way of recording and communicating their thoughts about classifying things. The abilities to classify, to reason about classification, and to extract information from a classification scheme are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. Ordinarily, people are not expected to deal with such problems until their command of their natural language is fully developed and until they face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires special modes of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification, thus freeing young minds (in some cases as young as four years of age!) to think logically and creatively about classes of things and to report their thinking long before they have the verbal skills that were thought to be required.

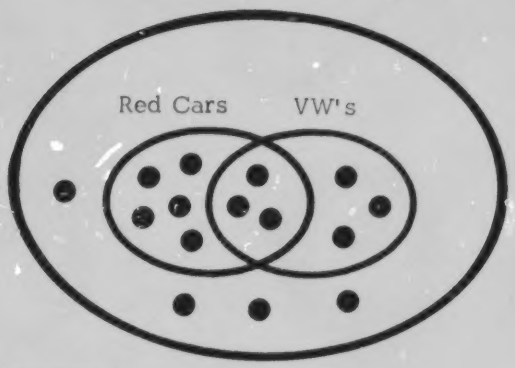
In this strand we set up situations and ask carefully phrased questions that continue to develop skills in classification, always remembering that the skills grow out of such experiences but that an ability to draw strings and arrows is not an objective in itself; the mode of thinking involved is the objective. Thus it is extremely important for us to set up our situations carefully. The objects we ask students to classify must have well defined attributes; otherwise, the students' attention will be distracted by the problem of deciding whether a certain object does or does not have a certain attribute. For example, consider

this situation. A teacher suggests that the blue string on the board is for all children in the class with dark hair and the red string is for all children in the class who are tall. She then invites a child, say John, to place a dot for himself on the diagram. He places a dot inside the blue string and outside the red string and labels it "J". There is a flurry of mixed agreement and disagreement in the classroom. John has brown hair and he thinks it is dark, but some say it is really light. John does not think he is tall, but someone insists that he is taller than most of the other children in the class. As a result, most of the energy in the class is directed toward making a decision about what attributes John has in this situation rather than how to classify John relative to those attributes. More care in choice of attributes would have avoided such distraction and confusion. For example, if the blue string were for all children in the class whose birthdays are in January and the red string for all the children who have a sister, John's choice of a dot for himself would be criticized strictly on its placement (i.e., his classification of himself) and not on his decision as to what attributes he has.

It is quite clear, then, why we make such extensive use of A-Blocks (attribute blocks) and sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, etc., and they know whether or not a given number of a set of numbers is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to be able to count the objects that have a certain attribute. For example, in a parking lot there are eight red cars and six Volkswagons. If there are four cars in the lot that are neither red nor Volkswagons, can we be sure there are 18 cars in the lot? Suppose we counted the cars in the parking lot and found there are exactly 15. Is this possible? A string picture immediately settles this apparent paradox.

Cars in Parking Lot



There are three red Volkswagons.

Relations: The Language of Arrows

In almost everything we do or think about we are intensely interested in the relations between things. This is true in our everyday lives, in school, in our careers, and in all scientific pursuits--we are always trying to establish, explore, and understand relations. Consider the following simple example. A girl comes running up to her mother and shouts: "Mommy, Mommy; Fido bit the mailman!" The girl is reporting to her mother that a certain relation has suddenly been established between the family dog and a certain mailman. Fido bit the mailman. The existence of the "biting relation" between Fido and the mailman has many important practical consequences. From the mailman's point of view, it is presumably a painful one and may lead him to consult a physician. From the mother's point of view, it is at the very least embarrassing, and might even lead to a potential law suit; from Fido's point of view, it will likely lead to punishment of some sort. For all these reasons and many others, the girl feels it urgent for the mother to know of the existence of the "biting relation" between the dog and the mailman.

To show our keen interest in relations and the great variety of relations that can exist between things, consider the following short list of everyday facts or statements. Each sentence asserts a relation between one object (person or thing) x and another y.

1. Johnny is taller than Mary. (establishes a relation between Johnny and Mary in terms of their height)
2. Johnny is Mary's brother. (establishes a blood relation between the two)
3. Johnny sent Mary a valentine.
4. Johnny borrowed $1 from Mary.
5. Johnny hates spinach. (a very common relation between a child and a vegetable)
6. Tootsie Rolls are 2¢ cheaper at Store A than at Store B. (establishes a relation between Store A and Store B in terms of their respective prices for Tootsie Rolls)
7. Giving is more blessed than receiving. (This example indicates that there are relations among relations. A certain relation (giving) is more blessed than another relation (receiving).)

Consider also the following sentences taken from various school subjects. Each sentence talks about a certain relation.

1. Mark Twain is the author of Tom Sawyer.
2. Shakespeare lived before Hemingway.
3. Huckleberry Finn is a sequel to Tom Sawyer.

Science:

1. Lead is heavier than aluminum.
2. Aardvarks eat ants.
3. Phobos is a moon of Mars.
4. Malaria responds to quinine. (a relation between a disease and a certain drug)

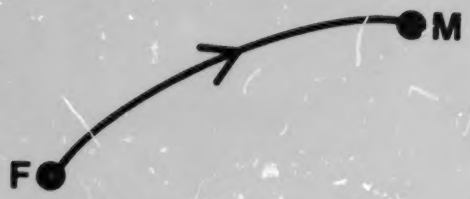
History:

1. Brutus killed Julius Caesar.
2. Louis XIV was King of France. (a relation between a person, Louis XIV, and a country, France)
3. Hitler hated Jews. (a relation between a person, Hitler, and a class of people, the Jews)
4. President Kennedy was assassinated in Dallas. (a relation between a person and a city)

Let's hope that the examples above have made our main point: we are interested in the relations between things. Mathematics is no different from everyday life, English literature, science, or history. To study mathematics means largely to study the relations between mathematical objects.

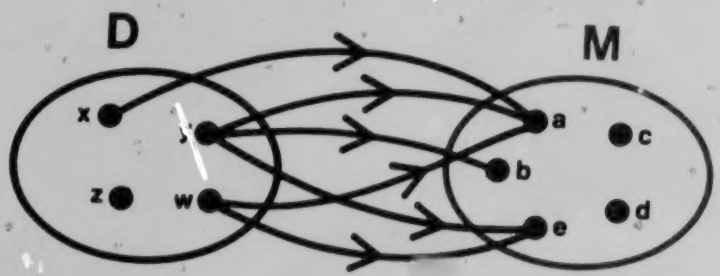
As always, if we are to study anything seriously, we will need a language, a way of symbolizing the things we want to study. A very apt language indeed for studying and talking about relations is the language of arrows. This is the language that is exploited systematically throughout this strand for thinking and writing about relations. Let's quickly remind ourselves how the language of arrows works. We have already agreed earlier to represent objects by dots. In the story of Fido biting the mailman, we would represent both Fido (F) and the mailman (M) by dots.

What could now be more natural to represent the fact that Fido bit the mailman than an arrow from Fido to the mailman.



In fact, with a little imagination the arrow suggests to our mind's eye quite vivid y Fido's rushing towards the mailman and biting him. Note that, of course, we have drawn the arrow fro do to the mailman. The reverse arrow (from the mailman to Fido) would mean the mailman bit Fido (in the spirit of the classical "man bites dog" saying!).

In general, if we are trying to picture any relation R between objects (biting, being the brother of, sending a valentine to, borrowing $1 from, and so on) we draw an arrow from object x to object y whenever x relates in the specified way to y. Suppose, for example, D is a set of dogs and M a set of mailmen, and the red arrows represent the relation: bites.



Note how very vividly our arrow diagram presents the state of affairs. The answers to the following kinds of questions can be read off from the diagram almost instantly.

Which dog seems to be the most vicious?

Which dog seems to be the gentlest?

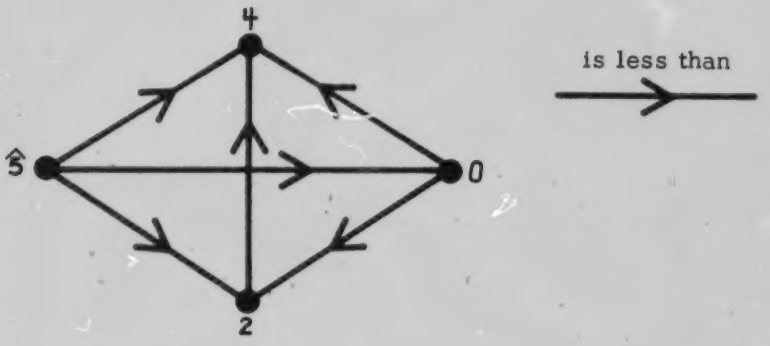
Which mailmen seem to get along best with the dogs?

Of course, frequently we are interested in comparing and contrasting two or more relations simultaneously. In that case we simply draw two superimposed arrow diagrams and code the arrows by color to show which arrows go with which relation.

In summary: Arrow diagrams are a very handy graphical representation of a relation, somewhat the same way that a blueprint is a very handy graphical representation of a house. By means of arrow diagrams we can represent important facts about a given relation or relations between objects in a very simple, suggestive, pictorial way--usually much more conveniently than the same information could be presented in words. The convenience of arrow diagrams has of course important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read"--and also draw--an arrow diagram of a relation long before he or she could read or present the same information about that relation in words or other symbols. The situation here is the same as in so many other cases. The problem of presenting certain ideas to children lies not in their intellectual ability to grasp the ideas; rather, the limitations are often just mechanical, motor problems of writing or reading about the ideas in conventional systems. The arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw.

In the discussion above, we introduced and explained the arrow diagrams in terms of a relation that comes from everyday life: dog bites mailman. Obviously, in the mathematics class we are also interested in relations between mathematical objects like numbers, lines, triangles, outcomes of experiments, and so on. However, we showed how arrow diagrams can be used to picture nonmathematical relations just as well as mathematical ones--for an important reason: to show that the mathematics class is not an isolated world in which we play exotic games totally different from what we do in the rest of our lives.

The important point here is simply this: thinking about and exploring the world of numbers is an activity very similar to thinking about and exploring any other world, e.g., the world of dogs and mailmen. Moreover, the tools we use to explore and understand the everyday world (e.g., arrow diagrams) are useful in exploring and understanding the world of mathematics. Conversely, the tools we develop and use to help us think about mathematical things very often help us in thinking about other kinds of things also. To see the analogies, consider the following simple arrow picture, where a red arrow from x to y means x is less than y.



To make our point more graphic, we imagine the numbers (which are, after all, things) talking to each other via the red arrows and saying: I am less than you.*

Now we can ask such questions as:

1. Who does the most talking (i.e., which is the smallest number)?

2. Who does the most listening (i.e., which is the largest number)?

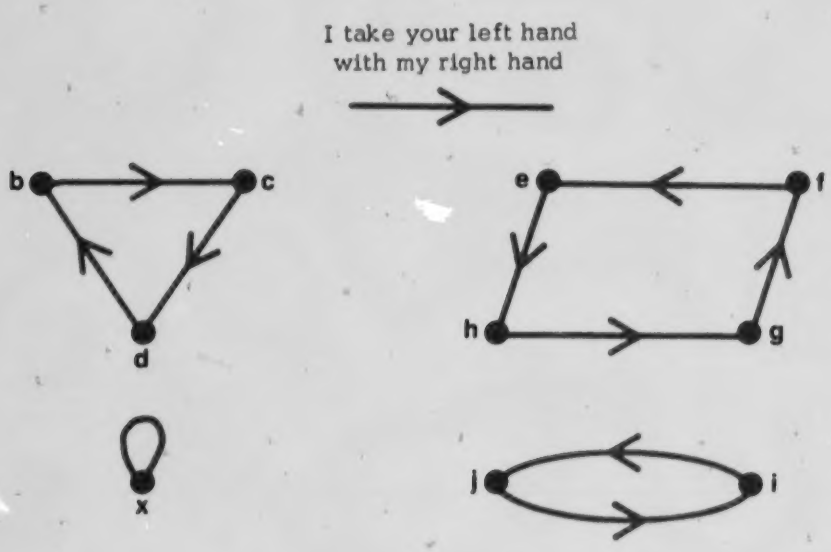
We leave it for the reader to find questions analogous to (1) and (2) above in the dogs and mailmen situation discussed previously.

---

* Aesop, to make his points, used talking wolves and foxes and dogs. In the same spirit, we like to have the numbers talk. After all, the number 2 probably knows more about its relations to other numbers than anyone else, so why not have it tell us what it knows?

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations between things. Again it must be remembered that our goals concern the thinking process and not the mechanism. Thus we must resist making the ability to draw prescribed arrows an objective in itself, just as we must resist viewing an arrow diagram as just another format for drill problems in arithmetic.

There is another educational bonus that occurs when arrow diagrams are used imaginatively. A story can be captured and described by an arrow diagram, and conversely, an arrow diagram can suggest many possible stories and games, allowing children to expand their creativity in many ways. In this sense, an arrow diagram resembles a script for a play or a choreographer's dance diagram. For example, suppose each child in the class is told to do the following: With your right hand take the left hand of one child. What confusion! When the instructions are eventually carried out they can easily make a record of this event (or it might even turn out to be a dance) with an arrow diagram that might look like this:

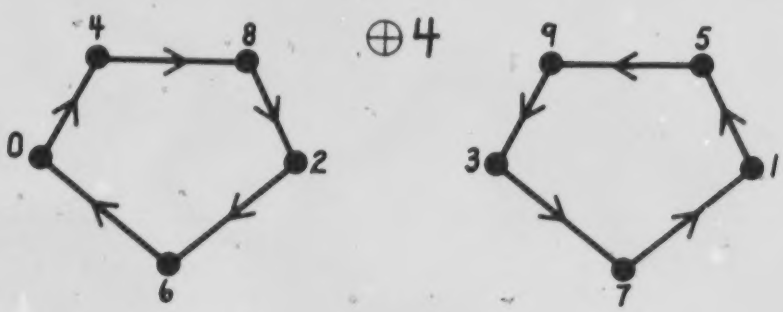


(Notice that student x took the easy way; he took his own left hand with his right hand.)

Conversely, given such an arrow diagram (which describes a relation called a permutation, because every dot has exactly one arrow starting from it and exactly one arrow ending at it), the children can use it as a "dance plan" and follow its directions in the next "dance". Of course, the same diagram might suggest a wide variety of stories that children can tell, such as:

> "On Christmas Day, everyone in our big family gave exactly one gift to someone in the family and received exactly one gift from someone (not necessarily the same person to whom his gift was given) in the family. Al (denoted "x" in the diagram) knew what he wanted, so he gave his gift to himself. Beth (b) gave a doll to Clara (c), ..."

It is not surprising that such arrow diagrams, and the games suggested by them, yield important mathematical concepts. Consider, for example, the following story situation in which the numbers 0, 1, 2, ..., 9 become separated from the rest of the whole numbers and are looking for a game to play. They decide they cannot play their usual addition games, for that would require all the whole numbers to be present. So the ringleader, Zero, suggests a new "addition" game, $\oplus$, that they can play, in which $1 \oplus 7 = 8$, $3 \oplus 6 = 9$, $2 \oplus 8 = 0$, $7 \oplus 6 = 3$, etc. Now they can all play the game because they use only the last digit of the ordinary sum for this new kind of sum. "Let's try the $\oplus 4$ snake dance!" shout the numbers. And they do.



"Also, the ⊕ 6 snake dance!"

"The same dance, but in the opposite direction," they notice.

The arithmetic of the set { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 } determined by this new operation, ⊕, illustrates some very basic concepts in mathematics, but at this level it is just one more interesting experience that generates good mathematical thinking.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze given situations.

By this time your students should be very familiar and quite comfortable with both the language of strings and the language of arrows. The content of this strand for the present semester builds in a variety of ways on this fluency with the languages. Each of the two chapters included in this strand present situations quite distinct from the other.

Chapter 1 continues the work begun in the chapter on Relations, Functions, and Composition (Chapter 1 in The Languages of Strings and Arrows strand) from CSMP Mathematics for the Intermediate Grades, Part II. Here the internal structure of the language is exploited to pose situations involving the composition of relations. The situations in this chapter very often lend support to and continue the work begun in The World of Numbers strand on the composition of numerical relations.

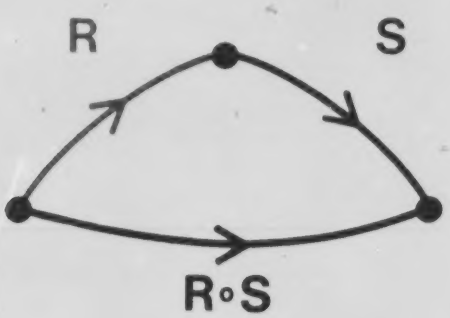
The lessons in Chapter 2 extend the String Game and logical thinking lessons from Part II (Chapter 2 in The Languages of Strings and Arrows strand) and continue a systematic analysis of a mathematically rich game (The String Game) which has become very familiar to the students.

1

RELATIONS, FUNCTIONS, AND COMPOSITION

In Parts I and II of the CSMP Intermediate Grades Program, the concept of composition of relations was explored through a series of games in which the relations were not specified but were identified only by colors. You may find it helpful to read the introductions to appropriate chapters from the Languages of Strings and Arrows strand in the teacher's guides for Parts I and II; Chapter 3, Composition Games on page 77 of Part I and Chapter 1, Relations, Functions, and Composition on page 1 of Part II. If necessary, you could borrow these teacher's guides from a fourth grade teacher.

The first lesson for this semester (Section 1.1) reviews the notion of composition of two unspecified relations with the relations denoted "R" and "S" and the following composites being considered: R • S, S • R, and R • R. After reviewing the idea of composition of relations, the lesson continues with some specific interpretations of the relations; for instance, some "family" relations such as "you are my father" and "you are my mother", and some numerical relations, such as +2, ×3, etc. In this manner, composition of relations takes on concrete meaning not only in an unspecified set of arrows but in familiar settings of family relations, arithmetic, etc. For example, if R is the relation, "you are my daughter" and S is the relation "you are my brother", then R • S is the relation "you are my son".



NOTE: "R • S" may be read, "R circle S" or "R followed by S" or "R then S". Some mathematicians write "S • R" for "R followed by S", and others write "R • S" as we do here. A choice has to be made; we selected what seems to be most natural for our purposes.

The fourth section of this chapter presents a lesson which uses various composite relations to present messages. This again extends similiar lessons on arrow messages in Part II of the CSMP Intermediate Grades Program (see Sections 1.4 and 1.5 of the Languages of Strings and Arrows strand of Part II).

The second and third lessons (Sections 1.2 and 1.3) review addition and multiplication modulo 10. These operations were introduced in lessons based on the storybook "Dancing Friends" which are part of the CSMP curriculum for the Upper Primary Grades, Part IV and for the Intermediate Grades, Entry Module I. They then continue to examine some interesting patterns generated by iterated composites of these relations in preparation for the lessons on the four-tailed kite (Sections 1.7 and 1.8).

The fifth lesson introduces an unspecified relation K in an arrow picture which resembles a kite. Iterated composites, K^2 (= K • K), K^3 (= K • K • K), K^4 (= K • K • K • K), and so on are explored and patterns relating these composites are revealed.

The sixth lesson extends the operations of addition and multiplication modulo 10 to addition and multiplication modulo 20, 30, 40, 50, and so on.

NOTE: In the lessons, operations of addition and multiplication modulo n are referred to as "addition and multiplication with n friends". For addition or multiplication with n friends, denoted "$+_n$" or "$\times_n$" respectively, the n whole numbers from 0 to n -1 are in the "game". To add some numbers modulo n add them in the regular fashion and then take only the remainder upon division of the sum by n, or subtract n as many times as is necessary until a number less than n remains. For example,

$7 +_{10} 8 = 5$ (because 7 + 8 = 15 and 15 - 10 = 5);

$23 +_{40} 31 = 14$ (because 23 + 31 = 54 and 54 - 40 = 14);

$10 \times_{30} 15 = 10$ (because 10 x 15 = 150 and 150 - (7 x 20) = 10).

The emphasis in this sixth lesson is to study addition and multiplication modulo 80, denoted "$+_{80}$" and "$\times_{80}$" respectively, because it is these operations which are used to create the four-tailed kite in Sections 1.7 and 1.8.

In the last two lessons in the chapter, a kite with four tails is constructed by building and connecting roads using the $2 \times_{80}$ relation. Ornamentation is added to the picture by considering the relationship between the $2 \times_{80}$ relation and the $+_{80}40$ relation. The end product of the construction is a full picture of the eighty whole numbers from 0 to 79 related by the $2 \times_{80}$ and $+_{80}$ 40 relations (see Language Poster #2 for IG-III).

CAPSULE LESSON SUMMARIES

Section 1.1: Given an arrow picture involving two unspecified relations and given rules for composition of the relations, draw all the arrows for certain composites. With various specifications for two relations R and S, interpret composites R • S and S • R.
(Worksheets L01, L02, L03)

Section 1.2: Review addition and multiplication modulo 10 using the games with ten friends introduced in the DANCING FRIENDS story. Practice calculating with these operations. Explore a pattern generated by repeatedly multiplying by 2 modulo 10.
(Worksheets L04, L05)

Section 1.3: Review addition and multiplication modulo 10. Draw an arrow picture for the relation $\otimes 2$ (multiplication by 2 modulo 10) and examine iterated composites of this relation.
(Worksheets L06, L07)

Section 1.4: Identify messages involving the relations "you are my mother" and "you are my teacher", and interpret various composites of these relations.
(Worksheets L08, L09, L10, L11)

Section 1.5: Introduce the relation K in an arrow picture resembling a kite. Draw arrow pictures for iterated composites K^2 (= K • K), K^3 (= K • K • K), K^4 (= K • K • K • K), and so on and discover that $K^6 = K^4$, $K^9 = K^5$, $K^{10} = K^6$, $K^{11} = K^7$, $K^{12} = K^8 = K^4$, and so on.
(Worksheets L12, L13)

Section 1.6: Review addition and multiplication modulo 10 and introduce addition and multiplication modulo 20, 40, and 80. Practice various calculations with these operations. Explore a pattern generated by repeatedly multiplying by 2 modulo 80.
(Worksheets L14, L15)

Section 1.7: Review addition and multiplication modulo various multiples of 10 and review the pattern generated in Section 1.6. Construct roads starting at 1, 3, 7, and 9 using the relation $2 \times_{80}$. Connect these roads in an arrow picture which resembles a kite with four tails.

Section 1.8: Review the construction of the four-tailed kite from Section 1.7 and extend the picture by adding other whole numbers from 0 to 79. Relate $2 \times_{80}$ and $+_{80}\,40$ and use this relationship to ornament the four-tailed kite picture. Reveal the complete picture of the eighty whole numbers 0 to 79 with the $2 \times_{80}$ and $+_{80}\,40$ relations.
(Worksheets L16, L17)

1.1 Composition of Relations

Exercise 1

The arrow picture on page 10 is a copy of Language Poster #1 for IG-III. Exercise 1 makes use of this picture in order to review the notion of composition of relations. According to your preference, you should either carefully copy the picture on the board before the lesson begins or display a copy of the poster by taping it on the board. A second copy of this poster is provided in case a replacement is necessary. If you use the poster, draw the green and black arrows with felt markers or crayons. If you draw the picture directly onto a chalkboard, replace black with any previously unused color chalk (white, yellow, etc.).

Trace several red arrows in the picture and point to "R".

T: The red arrows are for a relation we call "R".

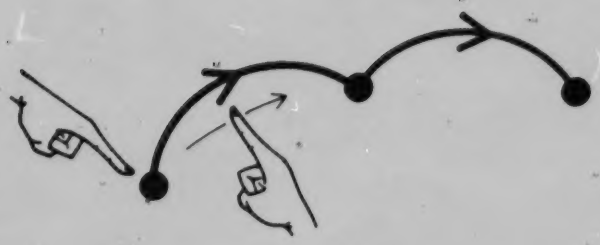
Then trace several blue arrows and point to "S".

T: The blue arrows are for a relation we call "S". We will use green arrows for the relation "R followed by S" (point to "R ∘ S"). Here is the rule for drawing green arrows.

Use the upper left corner of the arrow picture for your explanation. As you explain the rule stated in the left column below, make the motions described in the right column.

T: Each time there is a dot from which a red arrow starts ...	Place your left forefinger on a dot at which a red arrow starts. Hold your left finger there. With your right forefinger, trace the red arrow in the direction of the arrowhead.

1L



T: ... and the red arrow ends at a dot from which a blue arrow starts ...

Stop the motion of your right forefinger at the middle dot, tap the dot, and then trace the blue arrow.

T: ... then we must draw a green arrow from the dot where the red arrow starts ...

Tap this dot several times with your left forefinger.

... to the dot where the blue arrow ends.

Tap this dot several times with your right forefinger.

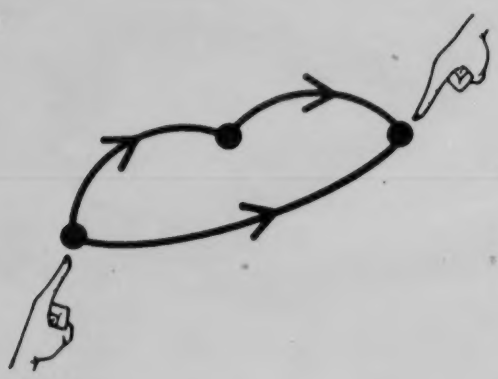
First trace and then draw the green arrow.

Repeat this series of motions, saying "red ... followed by blue ... is green."

T: There are many more green arrows we can draw in this picture. Who can see where another green arrow should be drawn?

Let the students take turns coming to the board and tracing where a green arrow should be added. If correct, invite the student to draw the green arrow.

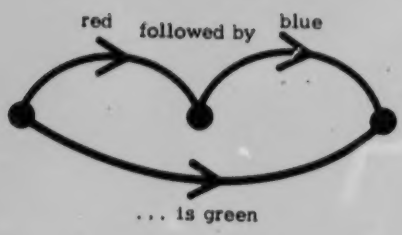
Encourage the students to point to the starting dot with their left forefinger, and then to trace the red arrow followed by the blue arrow with their right forefinger. Finally, their left and right forefingers should point respectively to the starting and ending dots for the green arrow.

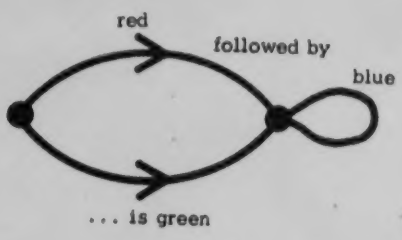


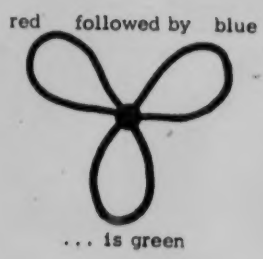
or

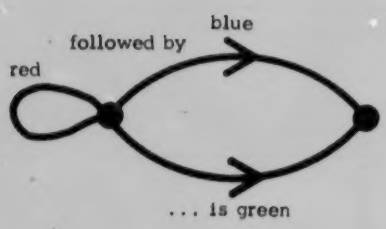
1L

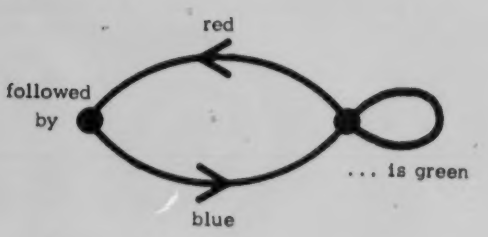
As they trace the arrows, the students should say "red followed by blue is green." The following diagrams show various ways green arrows may be found in the picture.











If a student has difficulty, you could ask one of the following questions:

T: Where should the green arrow start? (at a dot where a red arrow starts)

Where should the green arrow end? (at a dot where a blue arrow ends)

Did you trace <u>one</u> red arrow followed by <u>one</u> blue arrow?

When many green arrows have been found, write this at the top of the picture in black:

$S \circ R$

NOTE: Black is used in this guide for the fourth color. You may substitute another color if desired.

T: We will use black arrows for this relation (point to "S • R"). What do we call this relation?

S: "S followed by R" (or "S circle R" or "S then R").

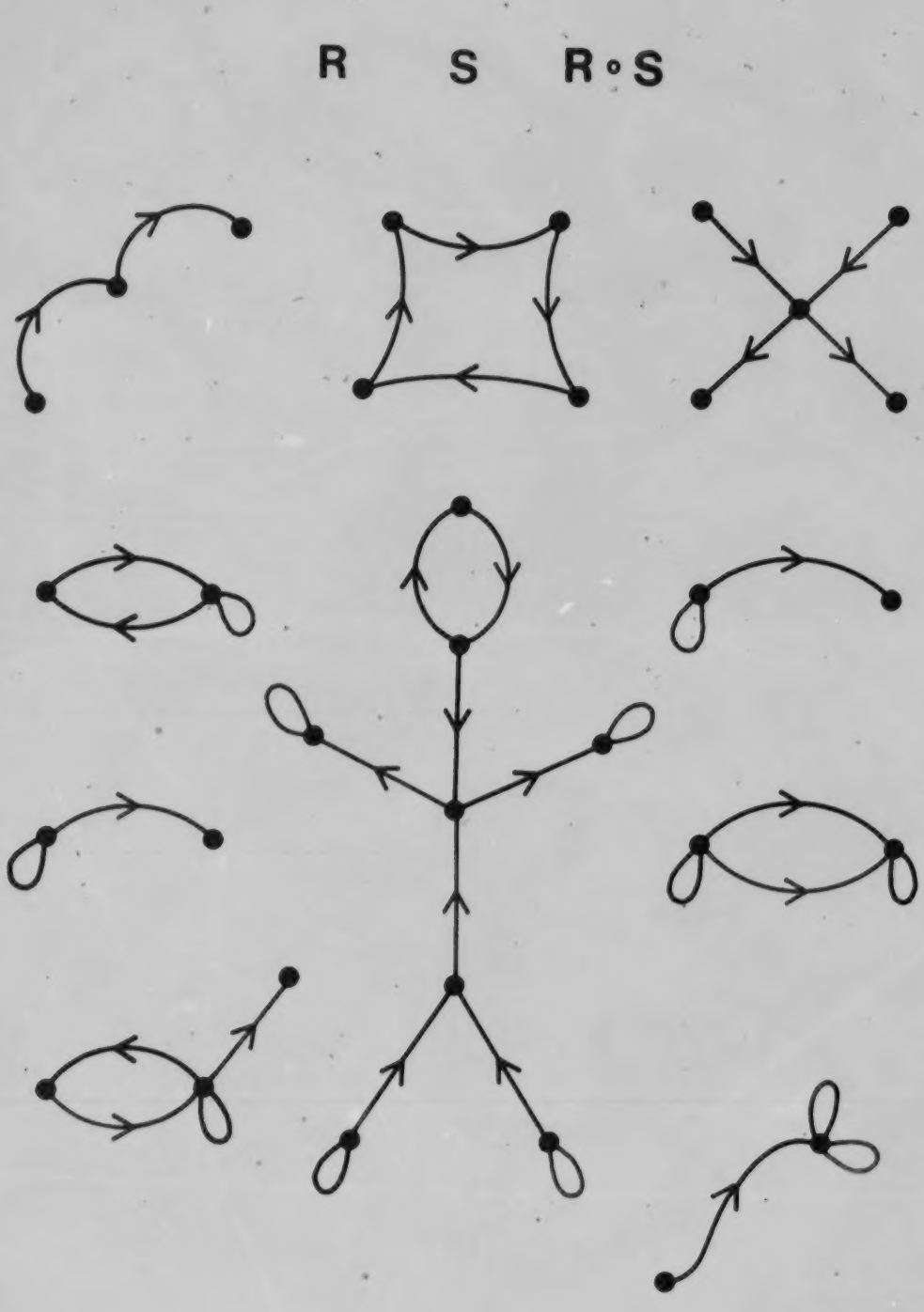
T: Where can a black arrow be drawn? Why?

S: Here is one. S circle R means that a blue arrow followed by a red arrow gives a black arrow.

S: Blue then red is black.

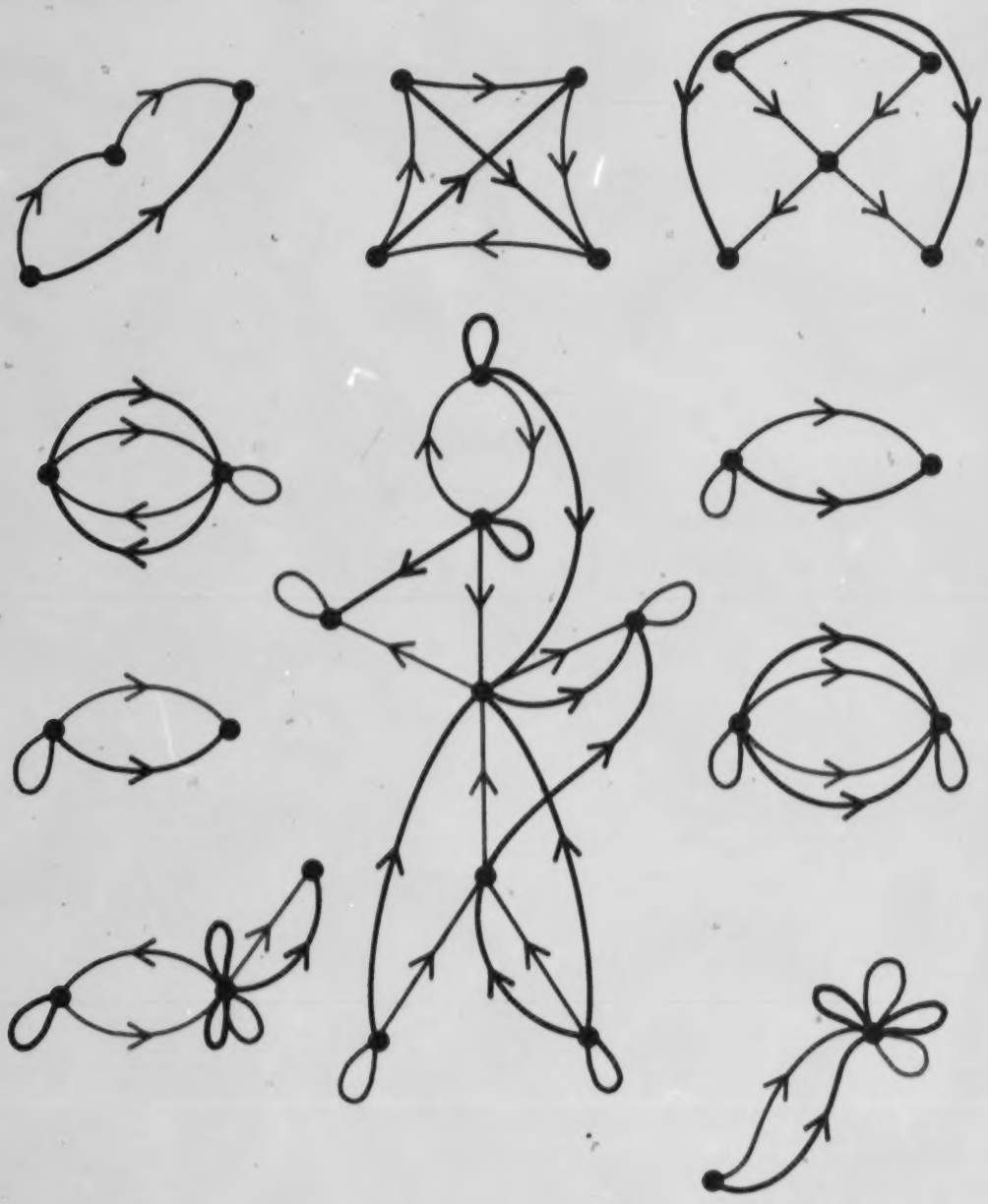
Invite the students to find more green and black arrows and to draw them in the picture. Every student should be able to participate in this activity, and it should move quite quickly. With a little encouragement, your class should be able to find all the green and black arrows. An answer key appears on page 11.

1L



Answer key for the green relation "R • S" and the black relation "S • R".

R S R∘S S∘R

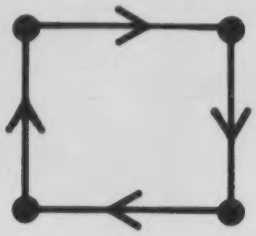


1L

Exercise 2

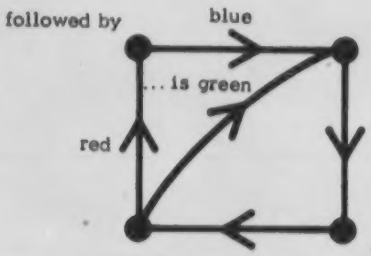
Draw this arrow picture on the board.

R S R∘S



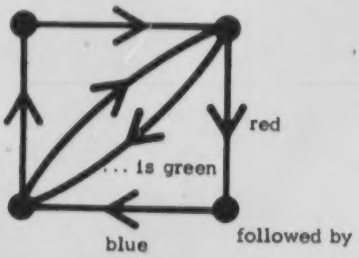
T: Can you draw a green arrow in this picture?

S: Red followed by blue is green.



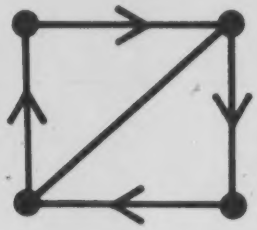
T: Very good. Can you find another green arrow?

S: There is one in the other direction.



T: Do you remember how we could draw this kind of picture in a simpler way?

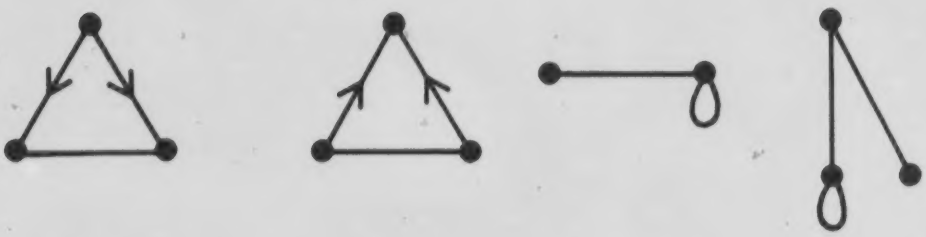
S: Use a green cord in place of the two green arrows.

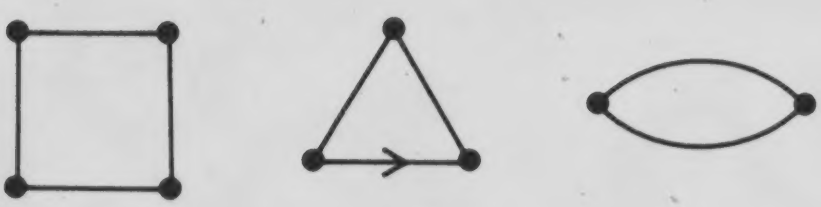


T: That's right. When we have the same color arrow in both directions between two dots, we can use a cord to represent the two arrows.

Draw this arrow picture on the board. (Black is used in this guide; however, you may substitute any fourth color you prefer.)

R S R∘S R∘R





Ask the students to begin looking for green and black arrows.

1L

T: How do we find where the green arrows can be drawn?

S: The same as before; red followed by blue is green.

T: What relation will the black arrows be for?

S: R circle R.

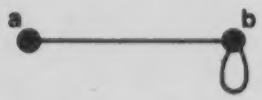
T: How do we find where black arrows can be drawn?

S: Red followed by red is black.

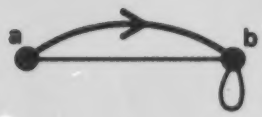
T: Who can draw a green arrow or a black arrow? First trace the arrow and tell the class which color arrow you are tracing; then if the class agrees you can draw the arrow.

Let the students take turns adding green arrows or black arrows to the picture. Encourage them to use similar language and tracing techniques as in Exercise 1. When there is one color arrow in both directions between two dots, the class should observe that it would be simpler to draw a cord. If necessary, mention it yourself.

For your reference, the following sequence shows some black arrows which may be difficult to discover.



This is one piece of the arrow picture with the dots labeled for easy reference.



Start at a. Trace the red cord to b, and then trace the red loop to end at b. Draw a black arrow from a to b.

Start at b. Trace the red loop, and then trace the red cord to end at a. Draw a black arrow from b to a.

The two black arrows can be replaced by a black cord.

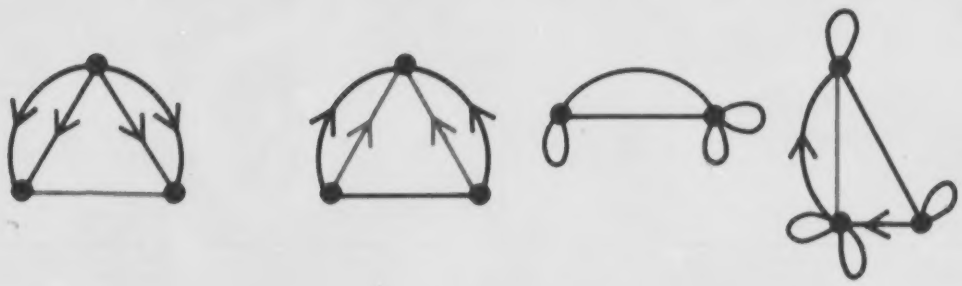
Start at b. Trace the red loop twice and end at b. Draw a black loop at b.

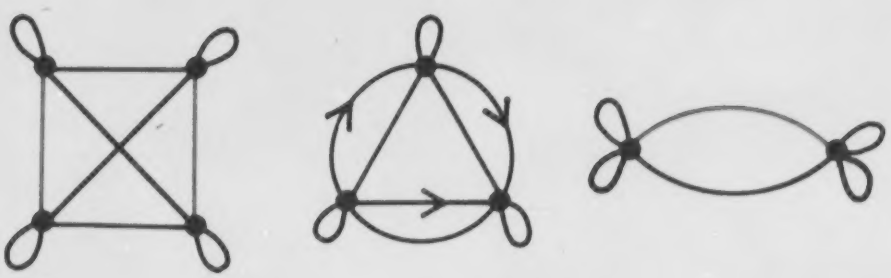
Start at a. Trace the red cord to b, and then trace the red cord to return to a. Draw a black loop at a.

The above sequence illustrates how two black arrows may be replaced by a cord, and how a red cord or loop can be followed twice to produce a black loop. If no student suggests this, you should mention that a loop or cord can be used twice.

Allow the exercise to continue until your picture looks like this answer key.

R S R∘S R∘R



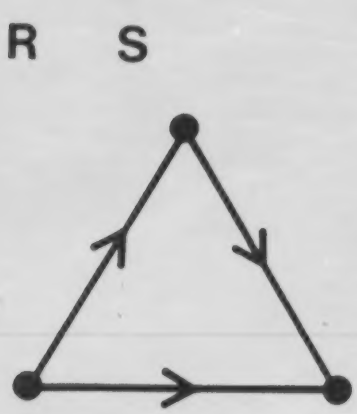


Erase the board before going on to Exercise 3.

1L

Exercise 3

Draw this arrow picture and table on the board.



R	S		

Point to the green arrow.

T: What relation could the green arrow be for?

S: R • S.

Label the third column of the table " **R ∘ S** ". Enter "+ 5" and "– 8" in the R and S columns, respectively.

R	S	R∘S	
+5	-8		

T: The dots are for numbers. The red arrow is for the relation "+ 5" and the blue arrow is for the relation "- 8". What could the relation "R • S" be?

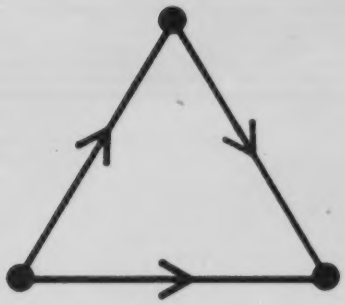
S: - 3, because + 5 followed by - 8 is - 3.

S: - 3, because if you add 5 to a number and then subtract 8, that is the same as subtracting 3.

Enter "- 3" in the R • S column.

Next enter "you are my father" and "you are my mother" in the R and S columns, respectively.

R S R • S



R	S	R • S	
+5	-8	-3	
you are my father	you are my mother		

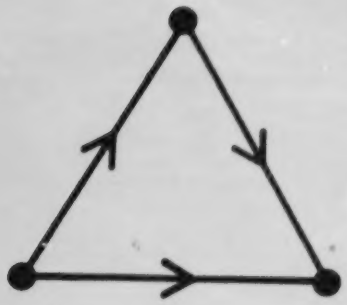
T: Now the dots are for people. The red arrow is for the relation "you are my father". The blue arrow is for the relation "you are my mother". What could the relation "R • S" be?

S: You are my paternal grandmother.

Enter "you are my paternal grandmother" in the R • S column. Complete the table by presenting R and S relations as indicated in the next illustration and asking the students to determine R • S.

1L

R S R ∘ S

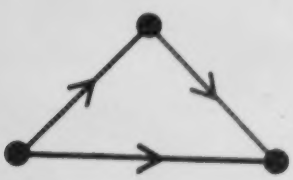


R	S	R ∘ S	
+5	-8	-3	
you are my father	you are my mother	you are my paternal grandmother	
-4	-7	-11	
2×	3×	6×	
you are in my math class	you are not in my math class	you are not in my math class	
4×	$\frac{1}{2}$×	2×	
you are my father	you are my sister	you are my father's sister	

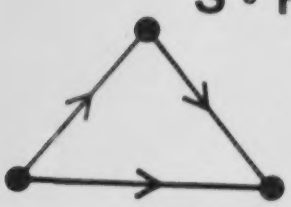
If any of these problems cause difficulty, you could label a dot with a number or a name and invite the students to label the other dots before deciding what the composite relation R • S could be.

Label the fourth column of the table "S • R", and draw a second arrow picture as illustrated. Invite students to complete the fourth columns by determining S • R for each pair of R and S relations. A completed table is given below.

R S R ∘ S



S ∘ R



R	S	R ∘ S	S ∘ R
+5	-8	-3	-3
you are my father	you are my mother	you are my paternal grandmother	you are my maternal grandfather
-4	-7	-11	-11
2×	3×	6×	6×
you are in my math class	you are not in my math class	you are not in my math class	you are not in my math class
4×	$\frac{1}{2}$×	2×	2×
you are my father	you are my sister	you are my father's sister	you are my father

T: What do you notice about R • S and S • R?

S: They are sometimes the same, but not always.

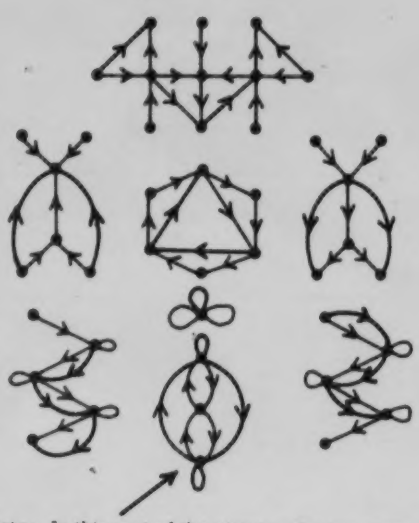
Lead a short discussion on R • S and S • R by comparing the third and fourth columns of the table.

Worksheets L01, L02, and L03 are available.

Name _____ L 01 *

Draw the missing green arrows and loops.

R S R∘S

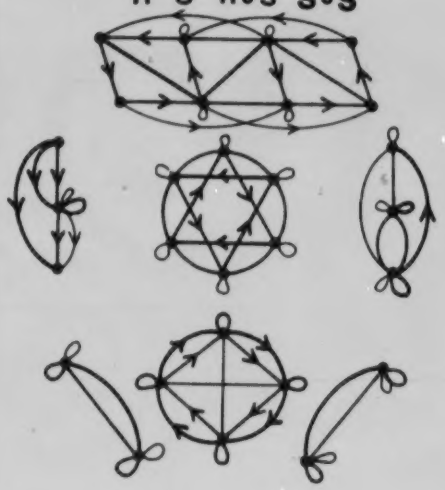


Note: In this part of the picture, the two green arrows may be replaced by a green cord.

Name _____ L 02 **

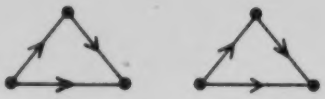
Draw the missing green and black arrows, loops, and cords.

R S R∘S S∘S



Name _____ L 03 ***

Complete the table.



R	S	R∘S	S∘R
-6	-13	-19	-19
5x	4x	20x	20x
3x	2x	6x	6x
+3	-7	-4	-4
8x	$\frac{1}{2}$x or +2	4x	$\frac{1}{2}$x or +2
you are my son	you are my sister	you are my daughter	you are my nephew
you are my mother	you are my father	you are my maternal grandfather	you are my paternal grandmother
you are my daughter	you are my husband	you are my son-in-law	you are my daughter
you are my friend	you are my brother	you are my friend's brother	you are my brother's friend
you are 5 years older than I	you are 3 years younger than I	you are 2 years older than I	you are 2 years older than I
you are older than I	you are the same age as I	you are older than I	you are older than I

1.2 Addition and Multiplication with Ten Friends

NOTE: This lesson explores some special calculations which were introduced in the storybook DANCING FRIENDS.* Two copies of this storybook are included in the classroom set of materials for this semester, and they should be made available to the students. Those students who were not in a CSMP class when this storybook was presented should be encouraged to read it sometime before this lesson; others may wish to read the story again. A short review of the story at the beginning of this lesson should be sufficient for most students to recall the special calculations with ten numbers.

Exercise 1

T: Do you remember the story of the DANCING FRIENDS?

Let the class recall as much of the story as they can. You may have to begin the discussion.

T: The boy in this story considers the numbers among his friends. He has invited ten number friends to his house to play some games. Do you know which numbers are playing and dancing together in the story?

S: Zero is the boy's best friend; he makes up most of the games.

T: That's right. Zero is the main character. Which other numbers are in the story?

S: All the whole numbers through 10.

---

* Lessons on the DANCING FRIENDS storybook are part of the CSMP curriculum for the Upper Primary Grades, Part IV and for the Intermediate Grades, Entry Module I.

S: 0 through 10 would be eleven numbers, and there are only ten.

T: Yes, the number 10 is not in the story; just the numbers 0 through 9. Do you remember the special kind of calculation the ten numbers used in their dances?

Write this problem on the board.

$$4 \oplus 8 =$$

T: They used this special plus sign. What is 4 ⊕ 8 (read: four circle plus eight)?

If a student answers "12", remind the class that only the numbers 0 through 9 were playing the game.

S: 2.*

T: Very good. Can you explain how you got 2?

S: 4 + 8 = 12 and I just kept the last digit, 2.

T: That's correct. When we see "⊕", we add in the usual way but only keep the last digit (or ones' digit). Let's try some more problems.

Write these problems, one at a time, on the board. Student responses are in the boxes.

$$8 \oplus 7 = \boxed{5}$$

$$5 \oplus 9 = \boxed{4}$$

$$2 \oplus 7 = \boxed{9}$$

Each time, ask a student to explain the answer.

---

* If someone in the class does not rather quickly give this answer, give it yourself and then continue with the other problems in a "Guess My Rule" spirit. As the pattern becomes clear, invite a student to describe the rule.

S: 8 + 7 = 15 and the ones' digit is 5.

S: 5 + 9 = 14 and we keep the 4.

S: 2 + 7 = 9 and there is only one digit, so 2 ⊕ 7 = 9, also.

Put the following problems on the board and ask the students to solve them on their papers. As the students are working, observe some of their answers. If some students answer with numbers larger than 9, remind them that only the numbers 0 through 9 are playing in this game.

$6 \oplus 9 = \square$ (Answer: 5)

$4 \oplus 6 = \square$ (Answer: 0)

$9 \oplus 8 = \square$ (Answer: 7)

$9 \oplus 9 = \square$ (Answer: 8)

$\square \oplus 3 = 7$ (Answer: 4)

$\square \oplus 5 = 2$ (Answer: 7)

$7 \oplus 7 \oplus 7 = \square$ (Answer: 1)

$8 \oplus 7 \oplus 6 = \square$ (Answer: 1)

$9 \oplus 9 \oplus 9 \oplus 9 = \square$ (Answer: 6)

$5 \oplus \square \oplus 4 = 7$ (Answer: 8)

$1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9 = \square$ (Answer: 5)

3L

After the students have had some time to work by themselves, let different students write the answers on the board. For some of the more difficult problems, ask the students to explain the answer.

The explanation for the last problem might be to add all the numbers together to get 45 and then to take the ones' digit to get 5. That is a correct solution, but it would be interesting to point out the following way to do this calculation.

T: That's very good. You can add all the numbers in the usual way to get 45 and then just keep the ones' digit, so the answer is 5. There's a quicker way to do this calculation which involves a clever way of looking at the problem.

Indicate pairing 1 with 9.

$$1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9 = \square$$

T: What is $1 \oplus 9$?

S: 0.

T: Do you see other numbers we could pair to get 0's in this calculation?

S: 2 and 8; $2 \oplus 8 = 0$

S: 3 and 7; $3 \oplus 7 = 0$

S: 4 and 6; $4 \oplus 6 = 0$

Indicate these pairings on the board.

$$1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9 = \square$$

T: What is $1 \oplus 9 \oplus 2 \oplus 8 \oplus 3 \oplus 7 \oplus 4 \oplus 6$?

S: 0.

T: What is the result of this calculation?

S: 5, because $0 \oplus 5 = 5$.

Erase the board before going on to Exercise 2.

Exercise 2

T: These ten numbers also play a multiplication game.

Write this problem on the board and read it "three circle times four is?"

$$3 \otimes 4 =$$

S: 2; because $3 \times 4 = 12$ and we only keep the ones' digit.

Write the following problems on the board and ask the students to solve them on their papers. Observe individual students' work and give help where necessary.

$4 \otimes 6 = \square$ (Answer: 4)

$3 \otimes 9 = \square$ (Answer: 7)

$2 \otimes 3 = \square$ (Answer: 6)

$5 \otimes 4 = \square$ (Answer: 0)

$\square \otimes 3 = 8$ (Answer: 6)

$\square \otimes 2 = 4$ (Answer: 2 or 7)

$4 \otimes \square = 4$ (Answer: 1 or 6)

$8 \otimes \square = 2$ (Answer: 4 or 9)

3L

After the students have had some time to work by themselves, let different students write the answers on the board. For some of the more difficult problems, ask for explanations. The last three problems should suggest to the students that sometimes more than one solution is possible. When discussing these problems, be sure to ask the students to explain how they are sure that they have found all the solutions.

Erase the board before continuing with the next exercise.

Exercise 3

Progressively write these problems on the board and get student responses as indicated in the boxes.

$$2 \otimes 2 = \boxed{4}$$

$$2 \otimes 2 \otimes 2 = \boxed{8}$$

$$2 \otimes 2 \otimes 2 \otimes 2 = \boxed{6}$$

If a student responds "16", let the class correct the mistake and remind everyone that only the numbers 0 through 9 are playing the game.

$$2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 = \boxed{2}$$

T: It gets tiring to write all those "2"'s and all those "$\otimes$"'s. So we'll use a shorthand way of writing. Instead of "2 $\otimes$ 2", we write "2^2" and read this "two to the second power."

Write this information on the board.

$$2 \otimes 2 = 2^2 = 4$$

T: For "2 ⊗ 2 ⊗ 2" we write "2^3" and read this "two to the third power."

Write this information on the board.

$$2 \otimes 2 \otimes 2 = 2^3 = 8$$

T: Now I will rewrite all the calculations on the board in this new notation.

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 6$

$2^5 =$

T: What number is 2^5 ?

S: 2, because 6 ⊗ 2 = 2.

Continue in this way for the following calculations.

$2^6 = \boxed{4}$

$2^7 = \boxed{8}$

$2^8 = \boxed{6}$

$2^9 = \boxed{2}$

$2^{10} = \boxed{4}$

$2^{11} = \boxed{8}$

$2^{12} = \boxed{6}$

3L

T: How are you getting the answers so fast?

S: There's a pattern; they just repeat in order 2, 4, 8, 6; 2, 4, 8, 6; and so on.

You may find it helpful to highlight this pattern by organizing the calculations on the board in groups of four.

$2^1 = 2$ $2^5 = 2$ $2^9 = 2$

$2^2 = 4$ $2^6 = 4$ $2^{10} = 4$

$2^3 = 8$ $2^7 = 8$ $2^{11} = 8$

$2^4 = 6$ $2^8 = 6$ $2^{12} = 6$

T: Be careful, because now I'm going to skip around. What number is 2^{14}?

S: 4.

Write the calculation on the board and continue with other problems. For each problem, allow a minute for all the students to think about the solution.

$2^{14} = 4$

$2^{18} = \boxed{4}$

$2^{21} = \boxed{2}$

$2^{23} = \boxed{8}$

$2^{28} = \boxed{6}$

$2^{40} = \boxed{6}$

Some students may make guesses. For example, for the big jump to 2^{40} you might get several incorrect answers before the correct one.

T: What number is 2^{40} ?

S: 2.

T: No.

S: 8.

T: No.

S: 4.

T: No.

S: It must be 6; there's no other choice.

T: You're correct. But was it necessary to guess or was there a way to be sure it was 6 from the beginning?

S: I was sure it was 6, because 40 is a multiple of 4 and whenever it is 2 to a multiple of 4 power, the answer is 6.

Accept any reasonable explanation, but do not force the students to be too explicit if an explanation is not forthcoming. The explanation above is very sophisticated, and you should not expect it from any but the best students.

If students seem to be catching on to the pattern, continue with the following problems.

$2^{41} = \boxed{2}$

$2^{50} = \boxed{4}$

$2^{53} = \boxed{2}$

$2^{100} = \boxed{6}$

3L

$$2^{105} = \boxed{2}$$

$$2^{1,001} = \boxed{2}$$

NOTE: A quick way to find 2^n is the following:

Divide n by 4 and consider the remainder.

1) If the remainder is 0, then $2^n = 6$.
2) If the remainder is 1, then $2^n = 2$.
3) If the remainder is 2, then $2^n = 4$.
4) If the remainder is 3, then $2^n = 8$.

Do not expect the class to discover this pattern exactly, but encourage them to continue searching for a pattern. If some students suggest this pattern, praise their good thinking and let the others continue to think about it.

Worksheets L04 and L05 are available for individual work.

Name _____ L 04 *

Complete:

2 ⊕ 4 ⊕ 6 ⊕ 8 = 0

2 ⊗ 4 ⊗ 6 ⊗ 8 = 4

1 ⊕ 3 ⊕ 5 ⊕ 7 ⊕ 9 = 5

1 ⊗ 3 ⊗ 5 ⊗ 7 ⊗ 9 = 5

3 ⊕ 4 ⊕ 5 ⊕ 6 = 8

3 ⊗ 4 ⊗ 5 ⊗ 6 = 0

The number in the box could be:

□ ⊗ 7 = 3 — 9.

4 ⊗ □ = 2 — 3 or 8.

5 ⊗ □ = 5 — 1 or 3 or 5 or 7 or 9.

Name _____ L 05 **

$3^1 = 3$

$3^2 = 3 \otimes 3 = 9$

$3^3 = 3 \otimes 3 \otimes 3 = 7$

$3^4 = 3 \otimes 3 \otimes 3 \otimes 3 = 1$

$3^5 = 3$ $3^6 = 3$

$3^7 = 9$ $3^8 = 9$

$3^9 = 7$ $3^{10} = 7$

$3^{11} = 1$ $3^{12} = 1$

$3^{13} = 3$

$3^{14} = 7$

$3^{15} = 1$

57

4 L

1.3 Powers of ⊗2

T: Do you remember the special calculations that were invented in the DANCING FRIENDS story?

Let the class discuss the story and recall as much as they can about the addition and multiplication games invented for ten numbers. Review some calculations; for example,

$5 \oplus 8 = 3$ $\qquad$ $2 \otimes 4 = 8$

$3 \oplus 2 = 5$ $\qquad$ $7 \otimes 6 = 2$

$4 \oplus 6 = 0$ $\qquad$ $4 \otimes 9 = 6$

$6 \oplus 4 \oplus 7 = 7$ $\qquad$ $3 \otimes 2 \otimes 4 = 4$

Let students explain how to do these calculations and, if necessary, remind the class that only the numbers 0 through 9 are playing in these games.

Draw the following picture on the board using white (or your regular) chalk for the ⊗2 arrows.

NOTE: Try to arrange the dots in the same relative positions shown here; then the resulting arrow picture will better suggest the properties of this function.

T: How many dots are there in this picture?

S: Ten.

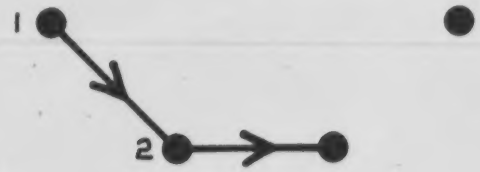
T: They are for the ten numbers, 0 through 9, in our game. The white arrows are for ⊗2.

Point to the dot at which the arrow ends.

T: Which number is here?

S: 2.

Label the dot "2". Draw this arrow starting at 2.



⊗2

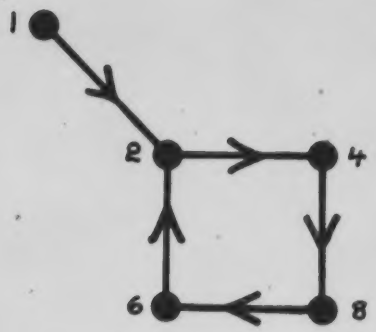
Point to the dot at which this arrow ends.

T: Which number is here?

S: 4, because 2 ⊗ 2 = 4.

Label the dot "4", and continue in this manner until your picture looks like the one in the next illustration.

4 L

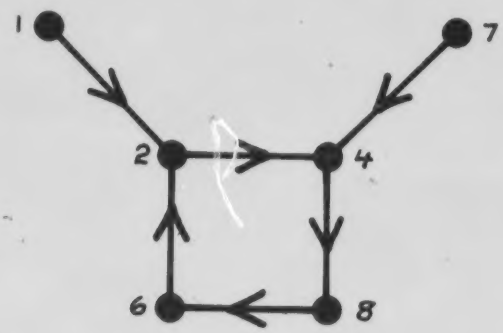


⊗2

T: Now we have arrows starting at five of the numbers. Is there an arrow that starts at 7?

S: Yes, it ends at 4.

Add this information to your picture.

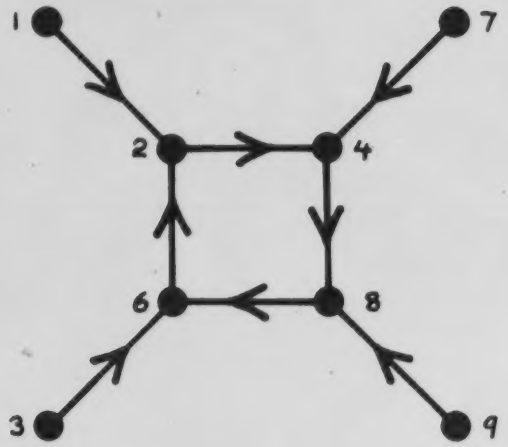


⊗2

T: Where should 3 be in this picture?
Where should 9 be?

S: There is an arrow starting at 3 and ending at 6, because 3 ⊗ 2 = 6.
There is an arrow starting at 9 and ending at 8, because 9 ⊗ 2 = 8.

Add this information to your picture.



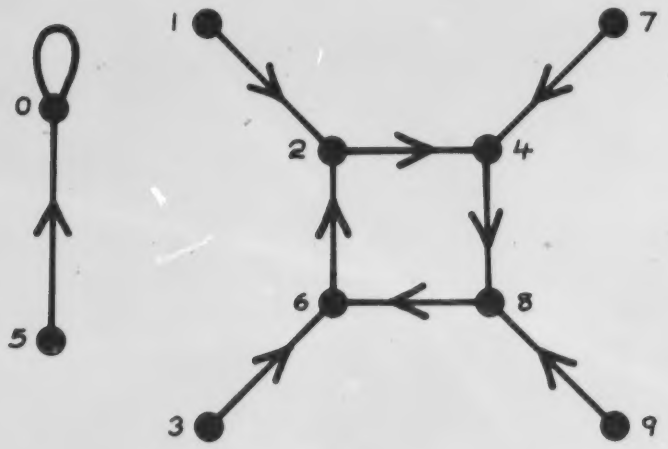
⊗2

Point to the two dots at the left.

T: Which numbers are these dots for?

S: 0 and 5.

Label the upper dot "0" and the lower one "5". Ask a student to draw all the missing arrows. Your picture should then look like this.



⊗2

4 L

T: Is ⊗2 a function?

S: Yes, there is at most one arrow starting at each dot.

T: How many arrows end at each dot?

S: Either two or none.

T: Which numbers have two arrows ending at them?

S: 0, 2, 4, 6, and 8; the even numbers in the game.

T: Let's call this function "R".

Indicate this by writing:

$$R = \otimes 2$$

T: Now, let's draw red arrows for R followed by R.

Add this information to the key.

$$R = \otimes 2$$
$$R \circ R = R^2$$

T: We can read this: R circle R or R followed by R or R to the second power or just R-two.
How do we know where to draw red arrows?

S: When there is a white arrow followed by another white arrow, then we can draw a red arrow.

If no student gives this or a similar explanation, you should suggest it yourself.

T: Is there a red arrow starting at 1?

S: Yes, it starts at 1 and ends at 4.

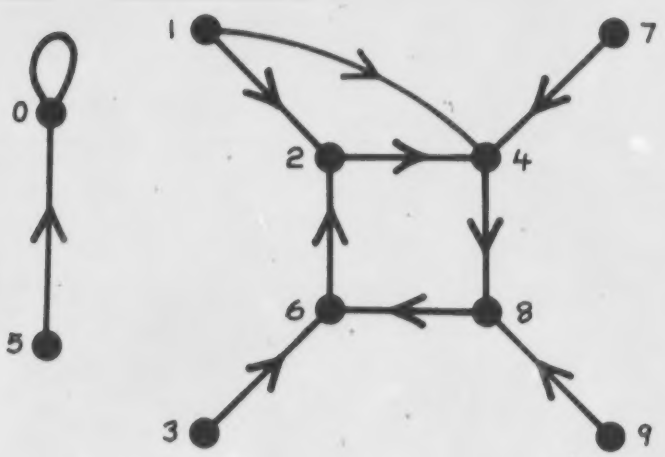
Demonstrate by putting your left forefinger at the dot for 1 and with your right forefinger, tracing the two white arrows ending at 4. As you do this say:

T: White followed by white . . .

Emphasize that you started at 1 (your left forefinger is there) and ended at 4 (your right forefinger is there). Then trace an arrow starting at 1 and ending at 4 as you say:

T: ...is red.

Draw the red arrow from 1 to 4.

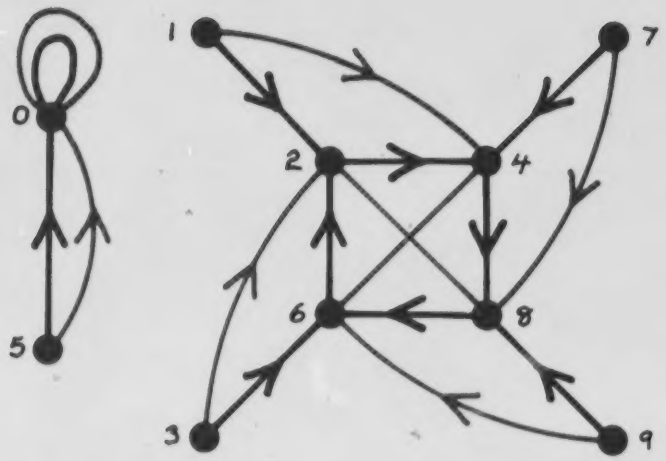


$R = \otimes 2$

$R \circ R = R^2$

Continue asking students to find and to draw red arrows. A student may suggest replacing "a round trip" by a red cord. If not, suggest it yourself in order to simplify the drawing. When all the red arrows have been drawn, your picture should look like the one in the next illustration.

4L



$R = \otimes 2$

$R \circ R = R^2$

T: Is R^2 a function?

S: Yes, there is just one red arrow starting at each dot.

Write this statement on the board.

$$R \circ R = R^2 = \otimes \square$$

T: Is there a number that could go in the box? Write your answer on your paper.

Walk around and check answers. After a minute or two, discuss this problem with the class.

T: Did you find a number that could go in the box? What function could R^2 be?

S: ⊗4.

T: How can we check if this is correct?

S: Check all the red arrows.

S: I did the calculations:

1 ⊗ 4 = 4
2 ⊗ 4 = 8
3 ⊗ 4 = 2
4 ⊗ 4 = 6
5 ⊗ 4 = 0
6 ⊗ 4 = 4
7 ⊗ 4 = 8
8 ⊗ 4 = 2
9 ⊗ 4 = 6

All the red arrows agree with these calculations, so ⊗4 is correct.

Distribute Worksheets L06 and L07.

T: On this worksheet (front and back) there are eight different copies of the relation R. First, color the " $\mathbb{R}^2$ " red, and in that picture draw the red arrows that we just drew on the board. You can copy them directly from the board. Also, put "4" in the box since we found that R^2 = ⊗4. Now you are to draw the arrows for R^3 on the copy in the upper right hand corner. Use any color you wish except red and black. Be sure to fill in the " $\mathbb{R}^3$ " in the color you choose. For an R^3 arrow, start at some dot and follow three black arrows in order to find the ending point. Then draw the arrow in the color of R^3. When you finish drawing all the R^3 arrows fill in the box next to "⊗". Then do the same thing with different colors for R^4, R^5, R^6, R^7, R^8, and R^9.

4L

As the students are working, walk around and check their drawings. If some students are having trouble, let them work with you collectively on R^3 while the others continue to work independently.

If there is time before the end of the lesson, stop those who have completed the L10 worksheet and begin the following discussion.

T: Did you find numbers that could go in the boxes? What are the functions R^3 and R^4 ?

S: $R^3 = \otimes 8$
$R^4 = \otimes 6$.

T: What about R^5 ?

S: $R^5 = \otimes 2$; its arrows are just like the black arrows so we didn't need to draw anything.

T: And R^6 ?

S: The R^6 arrows should be in red, because it's the same as R^2 .
$R^6 = R^3 = \otimes 4$.

S: And $R^7 = R^3 = \otimes 8$, so the arrows for R^7 should be the same color as those of R^3 .

S: And $R^8 = R^4 = \otimes 6$.

T: Do you see a pattern?

S: They keep repeating in fours like what we did last week.

T: What is R^{17} ?

S: The same as R or $\otimes 2$. Its arrows would be black.

Put these problems on the board and elicit student responses as given in the boxes.

$$R^{39} = \otimes \boxed{8}$$

$$R^{40} = \otimes \boxed{6}$$

$$R^{400} = \otimes \boxed{6}$$

T: Now here's a hard one. Think about it and tell me the answer tomorrow.

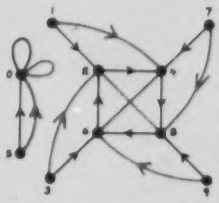
$$R^{2,379} = \otimes \boxed{}$$

NOTE: Since $2,379 = (4 \times 594) + 3$, $R^{2,379} = R^3 = \otimes 6$.

Name _____ L06 *

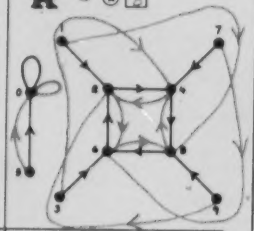
R = ⊗2

R^2 = ⊗ 4



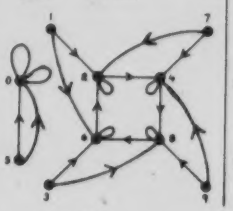
R = ⊗2

R^3 = ⊗ 8



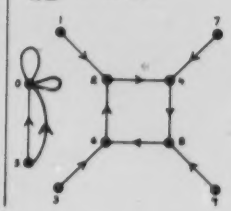
R = ⊗2

R^4 = ⊗ 6



R = ⊗2

R^5 = ⊗ 2

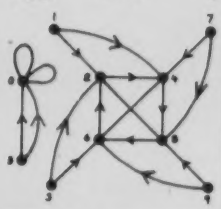


The choice of colors for these relations is left open to the student's preference; however, since $R^5 = R = \otimes 2$, R^5 should be in black.

Name _____ L07 **

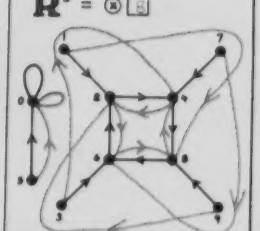
R = ⊗2

R^6 = ⊗ 4



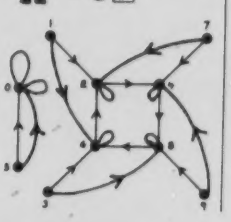
R = ⊗2

R^7 = ⊗ 8



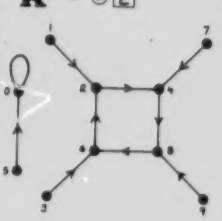
R = ⊗2

R^8 = ⊗ 6



R = ⊗2

R^9 = ⊗ 2



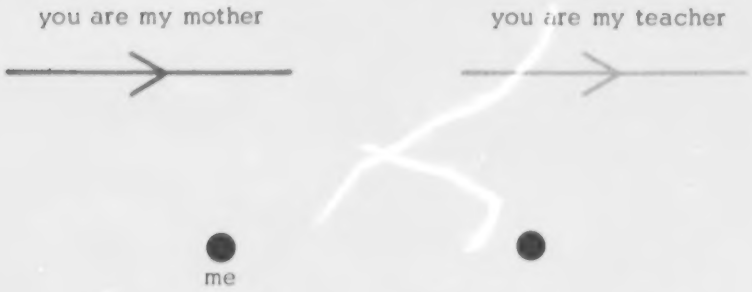
The colors here are determined by the choices on Worksheet L06, because $R^6 = R^2$, $R^7 = R^3$, $R^8 = R^4$, and $R^9 = R^5 = R$.

1.4 Arrow Messages

Before the lesson begins, be sure that each student has several pieces of paper and three different color pencils or crayons. Red, blue, and green are used in this lesson description, but you may substitute any three convenient colors.

Exercise 1

Put this information on the board.

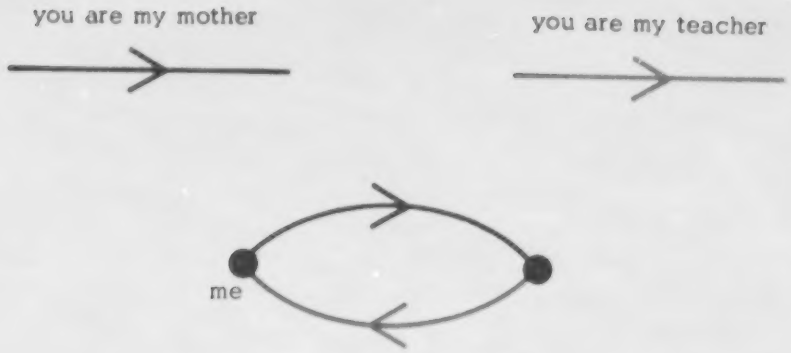


T: Red arrows are for the relation "you are my mother" and blue arrows are for the relation "you are my teacher". We can send messages with these two kinds of arrows. On your paper, draw this message with red and blue arrows.

I am my mother's teacher.

Write the message on the board as you read it. Allow the students to work independently for a minute or two. Then invite someone to draw the message on the board. You may have to remind the students that there are dots for only two people. After some discussion, the class should agree that the picture in the next illustration is correct.

6 L



I am my mother's teacher

Your questions should help the students to explain why the picture is for the given message.

T (pointing to the dot labeled "me"): This dot is for you, the speaker. Who is this other person (point to the unlabeled dot)?

S: My mother.

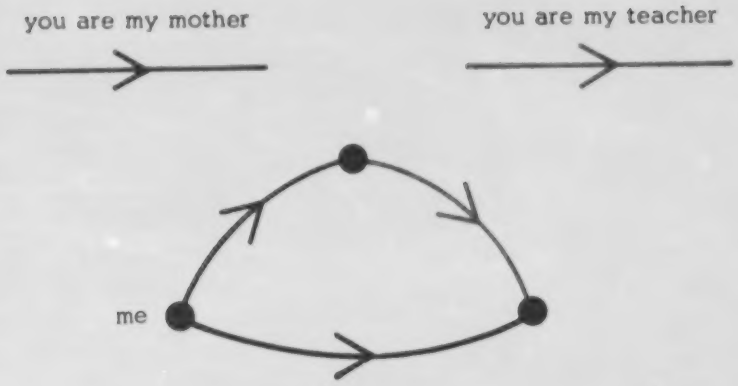
T: How do you know?

S: The red arrow is for "you are my mother", so I am pointing to my mother.

T: What does the blue arrow tell us?

S: My mother is pointing to me and saying: "You are my teacher." So I am my mother's teacher.

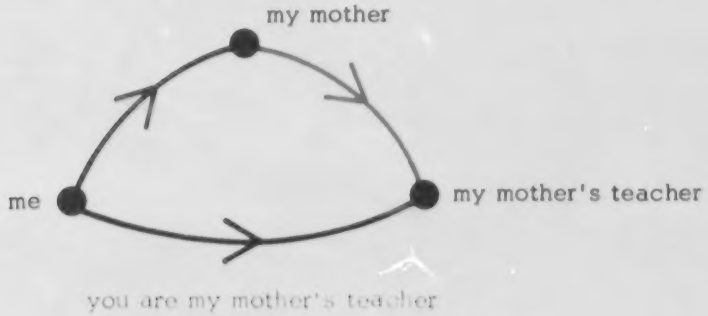
Erase the arrow picture, but leave the key arrows. Draw a new arrow picture as in the next illustration.



T: This arrow picture is for a different message. What could the green arrow be for? Write it on your paper.

As the students work on this problem, observe their answers. After a few minutes, call on someone to state the message and explain how to read the picture. You may have to ask appropriate questions to get a good explanation.

S: The green arrow is for "you are my mother's teacher". This dot (points to "me") is me and I point to my mother, so this dot at the top is my mother. My mother points to her teacher, so this dot (points to dot at the right) is my mother's teacher. I point with the green arrow to my mother's teacher.

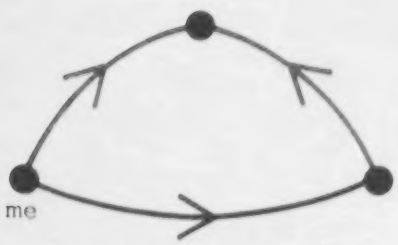


6 L

Continue this exercise by successively drawing the following arrow pictures on the board. Each time, ask what the green arrow could be for and, through appropriate questions, elicit a reading of the arrow picture. Encourage the students to be precise in their statements, but accept any messages that convey the correct relationships, however awkward.

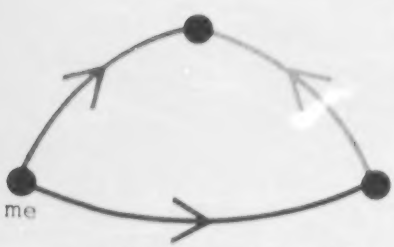
you are my teacher's mother

my teacher is your mother

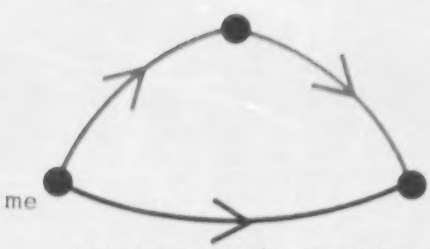


my mother is your mother
or
we have the same mother
or
you are my sibling (brother or sister)

your teacher is my teacher
or
we have the same teacher
or
you are my classmate

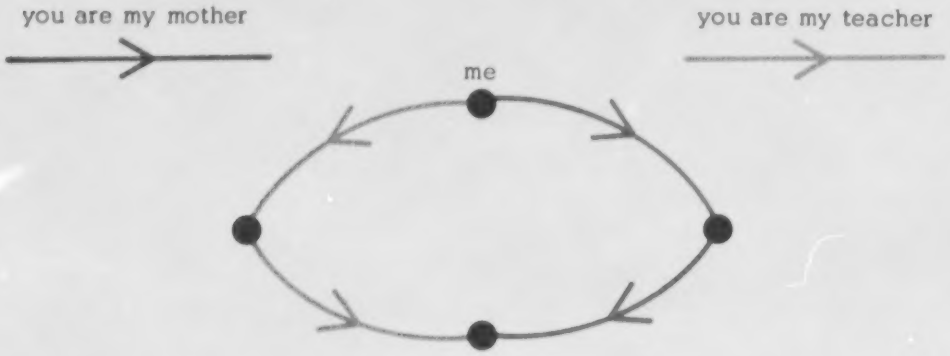


my mother is your teacher



you are my mother's mother
or
you are my maternal grandmother

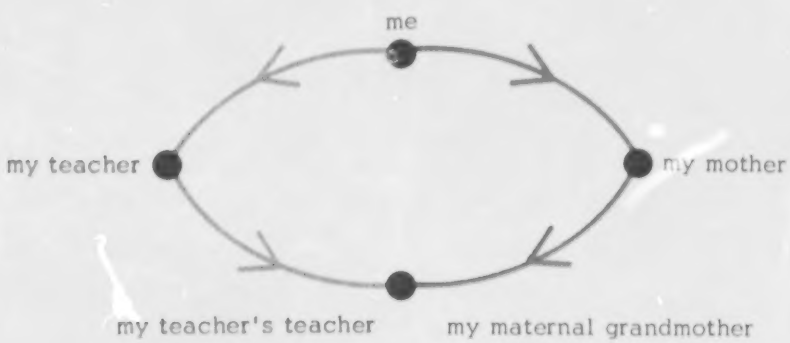
Erase the board except for the key arrows and draw this arrow picture.



T: This picture tells a message. Can you decode it? Write the message on your paper.

Give students several minutes to work independently. As you circulate, encourage students to write precise statements and ask leading questions of the students who have difficulty getting started. Ask a student to read the message and to explain how to decode the picture.

S: My teacher's teacher is my maternal grandmother. This is my teacher (points to the dot at the left) and this is my teacher's teacher (points to the dot at the bottom). This is my mother (points to the dot at the right) and this is my mother's mother or maternal grandmother (points to the dot at the bottom). So, my teacher's teacher is my maternal grandmother.

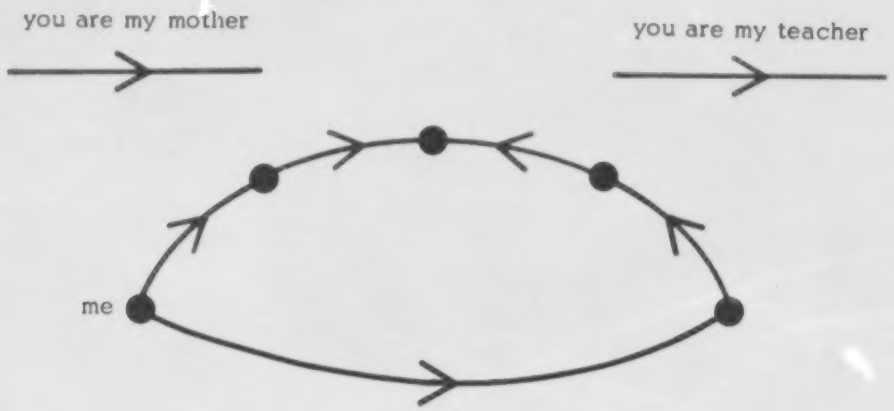


6 L

T: Are there other ways of saying this message?

S: My maternal grandmother teaches my teacher.

Erase the board and draw this picture.



T: There is a message in this picture. What could the green arrow be for? Write it on your paper.

Let the students have several minutes to think about this situation. Encourage them to write precise statements of what the green arrow could be for.

S: My mother's teacher is your mother's teacher.

S: Our mother's have the same teacher.

S: My mother and your mother are classmates.

Erase the board before going on to Exercise 2.

Exercise 2

Draw this arrow picture on the board.

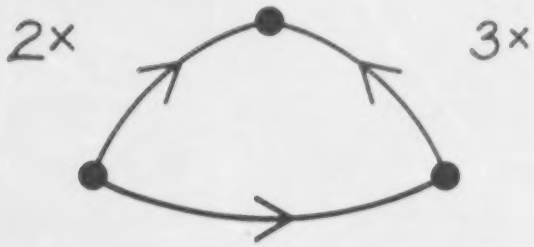
T: In this picture, the dots are for numbers and the red arrow is for +5 and the blue arrow is for +7. What could the green arrow be for?

S: -2.

A student might at first suggest that the green arrow could be for +12. Through the class discussion it should be noted that the directions of the arrows are important. Ask a student to explain why -2 is correct.

S: With the +5 arrow followed by against the +7 arrow is the same as +5 followed by -7 (the return of the +7 arrow), and $(+5) \circ (-7) = -2$.

Erase the keys "+5" and "+7", but not the arrow picture. Relabel the arrows as below.



6 L

T: Now, what could the green arrow be for?

Encourage the students to trace the arrows saying:

With red followed by against blue is green.

or

Red followed by return of blue is green.

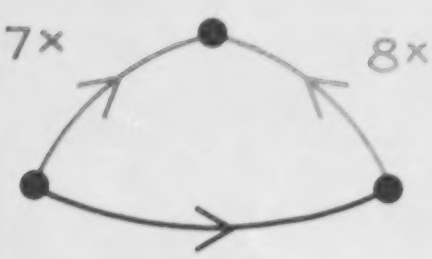
T: What is the return arrow for the blue arrow?

S: $\div 3$ (or $\frac{1}{3}x$).

T: So what could the green arrow be for?

S: $(2\times) \bullet (\div 3)$ or $\frac{2}{3}x$.

Repeat this exercise with the following arrow labels.



$\frac{7}{8}x$

$\frac{5}{4}x$

Worksheets L08, L09, L10, and L11 are available for independent work at the end of the lesson. Students who finish these worksheets quickly could create their own arrow pictures with messages to be decoded.

Name _____ L 08 *

You are my mother → You are my teacher →

Decode:

My mother is my teacher.

I am my mother's teacher.

I am the mother of my teacher.
or
My teacher is my child.

I am the teacher of my child.
or
I am the mother of my student.
or
My child is my student.

Name _____ L 09 **

You are my mother → You are my teacher →

What could the purple arrow be for ?

I am your maternal grandmother

I am your teacher's teacher

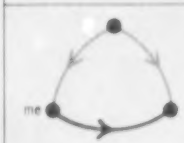
I am the mother of your teacher

I am the teacher of your mother

Name _____ L 10 ***

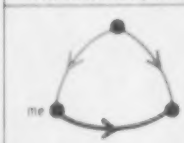
You are my mother → You are my teacher →

What could the purple arrow be for ?

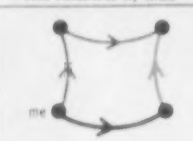


We are teachers of the same person
or
Your student is my student

You are my child's teacher
or
Your student is my child



You are the mother of my student
or
I am your child's teacher

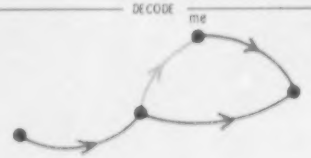


Your teacher is my maternal grandmother

Name _____ L 11 ****

You are my mother → You are my teacher →

DECODE



I am my sister's teacher

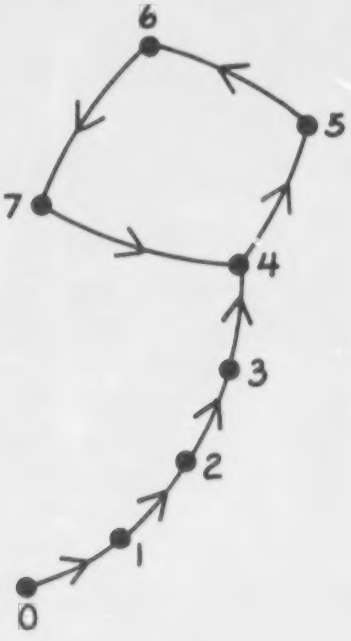
What could the purple arrow be for ?

My maternal grandmother is your mother's teacher

7 L

L1.5 The Kite

Draw the following arrow picture and chart on the board. Use white (or your regular) chalk for the arrows.



0	1	2	3	4	5	6	7

T: What does this picture look like?

S: A kite.

There may be many interesting suggestions, but recommend that it be called a "kite".

T: The arrows are for a relation we will call "K".

Write "K" with an arrow next to the chart as illustrated.

K

0	1	2	3	4	5	6	7

T: Look at the arrow starting at 0. Where does it end?

S: At 1.

Write "1" in the chart below "0".

T (pointing to the position in the chart below "1"): Which number should be here?

S: 2, because there is an arrow from 1 to 2.

Let the class tell you how to complete the chart for K.

K	0	1	2	3	4	5	6	7
	1	2	3	4	5	6	7	4

T: Is K a function?

S: Yes; there is at most one arrow starting at each number.

Below the chart for K, draw another chart for K ∘ K.

K∘K	0	1	2	3	4	5	6	7

Point to "K ∘ K" and ask:

T: What is this relation? How do we read it?

S: K followed by K (or K circle K).

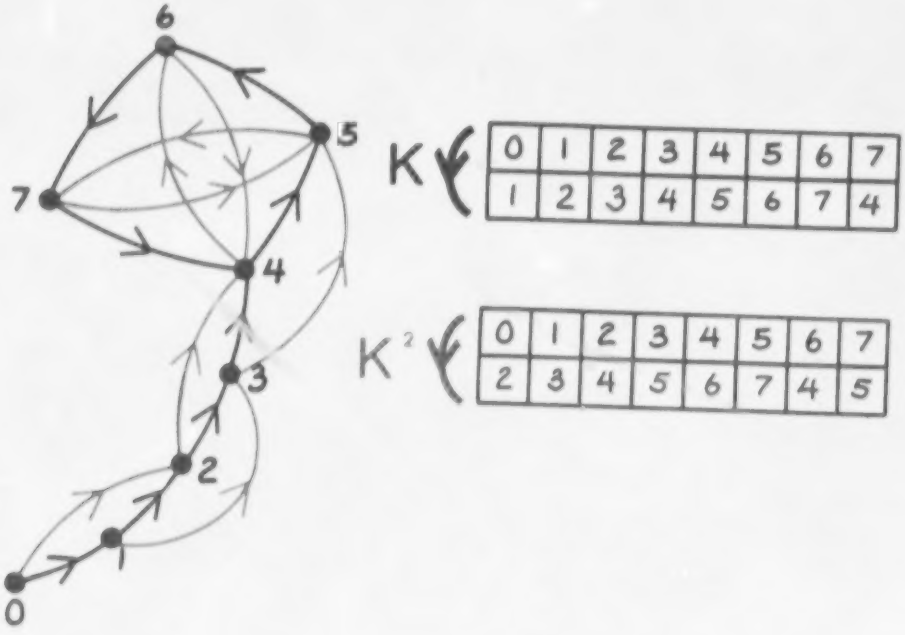
7 L

Write "$K^2 = K \circ K$" next to the chart as you say:

T: Sometimes we use this shorthand way of writing the K • K relation and call it "K to the second power" or simply "K-two". Where should we draw red arrows for K-two?

S: If we follow two white arrows in a row, then we can draw a red arrow. White followed by white is red.

With the motions described in LI.1, trace the white arrow from 0 to 1 and the white arrow from 1 to 2, and then trace the anticipated red arrow from 0 to 2. Ask a student to draw the red arrow from 0 to 2. Invite students to use the same motions for tracing arrows while saying "white followed by white is red." Then ask them to draw the resulting red arrows. Each time a correct red arrow is drawn, ask the student to fill in the appropriate space in the chart for K^2. A completed arrow picture and chart are given in the next illustration.



S: There is a round trip between 4 and 6 and between 5 and 7. We could use red cords between 4 and 6 and between 5 and 7.

Encourage this suggestion and simplify the picture by replacing the "round trip" arrows by red cords.

T: Is K^2 a function?

S: Yes, there is at most one red arrow starting at each dot.

Draw a third chart for K^3 below the other two.

$K^3 = K \circ K \circ K$

0	1	2	3	4	5	6	7

T: What relation will blue arrows be for?

S: White followed by white followed by white.

S: K to the third power or K-three.

Distribute Worksheets L12 and L13.

T: On these worksheets (front and back), there are eight copies of the kite. Arrows for K^2 are to be drawn in this picture (show the class Worksheet L12 pointing to the upper left picture) where there is a chart for K^2. You can copy the red arrows and the chart off the board. Then go on to the next picture (point to the upper right picture), draw blue arrows for K^2 and fill in the chart for K^3. When you finish K^3, go on to K^4 and so on.

7 L

Let the students work independently for about fifteen minutes. Help those who have difficulty getting started. As students complete Worksheet L12, ask them to go on to Worksheet L13 on the back. Tell the students they may choose any colors they like for the relations on L13, but that they should try to use different colors only for new relations and the same colors for relations which are the same. Students who finish both worksheets quickly could be given extra sheets of paper and encouraged to continue the activity for K^{10}, K^{11}, and so on.

While students are working independently, do not reveal the purpose of this part of the lesson which is to discover that $K^8 = K^4$, $K^9 = K^5$, $K^{10} = K^6$, $K^{11} = K^7$, $K^{12} = K^8 = K^4$, etc. Some students may observe these facts themselves; for others it will be pointed out during the last part of the lesson.

When you return to the collective lesson, ask students to put arrow pictures and charts for K^3 (blue), K^4 (green), and K^5 (purple) on the board. Ask students who have completed the second worksheet to put just the charts for K^6, K^7, K^8, and K^9 on the board. (Charts for K^{10} and K^{11} could also be put on the board if available.) Answer keys for these worksheets can be found at the end of this lesson description. We reproduce here just the charts.

K

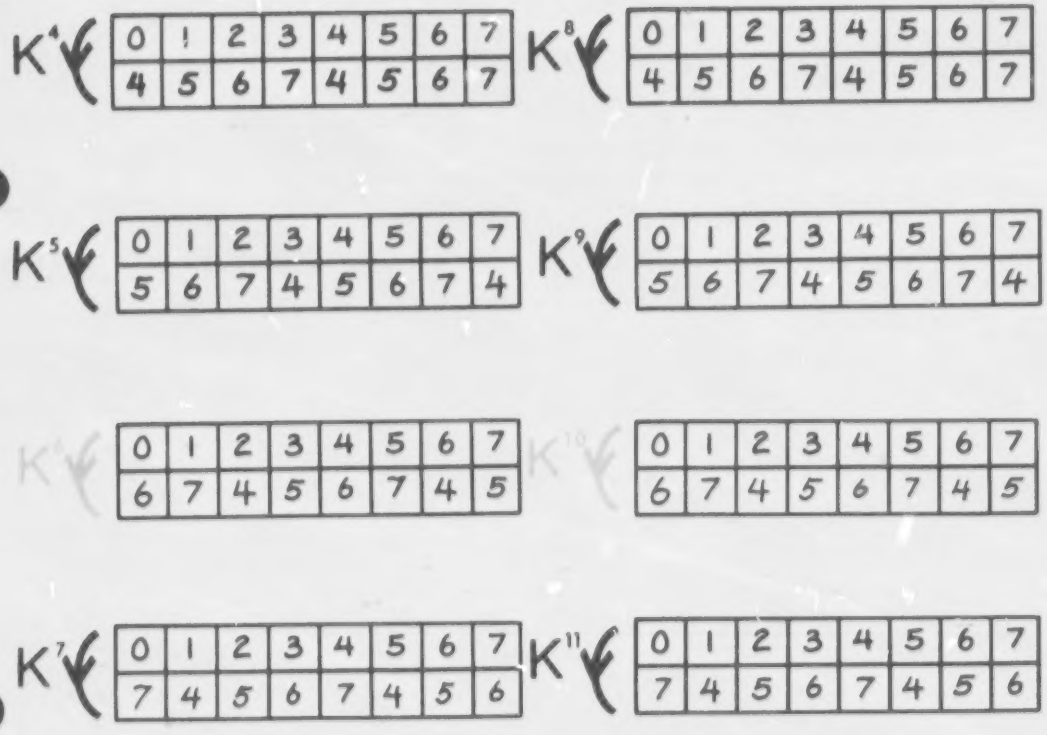
0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	4

K^2

0	1	2	3	4	5	6	7
2	3	4	5	6	7	4	5

K^3

0	1	2	3	4	5	6	7
3	4	5	6	7	4	5	6



T: What do you notice about the pictures and the charts?

S: After K^3 the only numbers in the bottom row are 4, 5, 6, and 7.

S: In the chart for K^4, they repeat in order, 4, 5, 6, 7; 4, 5, 6, 7.

S: K^4 and K^8 have the same chart; they have the same arrows.

If necessary, make a change of colors so that K^4 and K^8 are both in green and write the equality "$K^8 = K^4$" on the board.

S: $K^8 = K^9$ and $K^6 = K^{10}$ and $K^7 = K^{11}$.

If necessary, make appropriate changes in color and write these equalities on the board as they are mentioned.

7 L

T: Do you notice any other patterns in these charts?

S: Going from the chart for K^4 to the chart for K^8, the numbers 4, 5, 6, and 7 just shift one space to the left. The same going from K^6 to K^8 and K^8 to K^7.

Such a pattern may not be observed, or if it is may be difficult to describe. Accept any ideas your students have without much comment. If the class has observed the repeating pattern, you might continue the lesson by asking about other powers of K. The result of such further investigation would be to generate four strings of equalities.

$$K^4 = \quad K^8 = \quad K^{12} = \quad K^{16} = \ldots$$

$$K^5 = \quad K^9 = \quad K^{13} = \quad K^{17} = \ldots$$

$$K^6 = \quad K^{10} = \quad K^{14} = \quad K^{18} = \ldots$$

$$K^7 = \quad K^{11} = \quad K^{15} = \quad K^{19} = \ldots$$

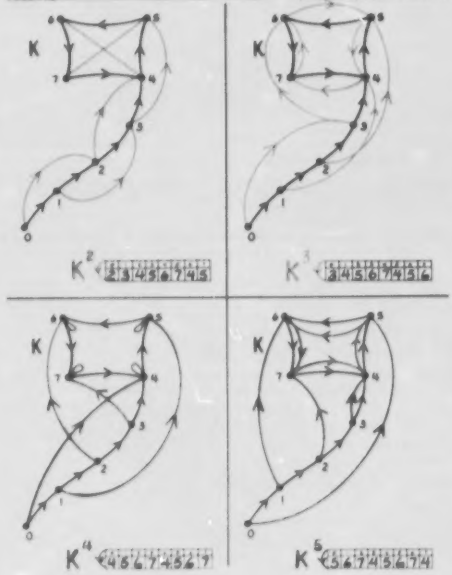
A final challenge question would then be:

T: What color should K^{103} be and why?

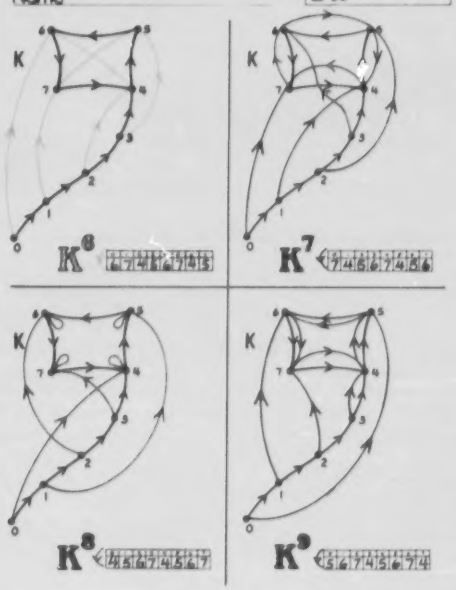
S: Orange, because K^{100} would be green just like K^4 and then following K three more times would give us the same as K^7.

You should not expect such a precise explanation. Accept any discussion which suggests this idea.

Name L12 *



Name L 13 **



9 L

1.6 Addition and Multiplication with Eighty Friends

Begin the lesson by briefly recalling the story of DANCING FRIENDS.

T: How many number friends were invited to play in the dancing games, and who were they?

S: Ten friends; the whole numbers 0 to 9.

T: Do you remember the special calculations the ten numbers used in their dances?

Write these problems on the board and ask for explanations from the students who solve them.

$7 \oplus 9 =$ $4 \oplus 8 =$

S: $7 \oplus 9 = 6$, because $7 + 9 = 16$ and we just keep the last digit.

S: $4 \oplus 8 = 2$, because $4 + 8 = 12$ and we keep the 2.

$4 \otimes 2 =$ $7 \otimes 8 =$

S: $4 \otimes 2 = 8$, because $4 \times 2 = 8$ and we keep the 8.

S: $7 \otimes 8 = 6$, because $7 \times 8 = 56$ and we keep the 6.

Exercise 1

T: Ten friends were in the Dancing Friends games. Another day ten more friends are invited to play in the games. How many number friends are there this day?

S: Twenty.

T: Can you guess which other number friends are invited?

S: The whole numbers 10 to 19.

T: That's right, the twenty friends playing this day are 0 to 19. They made up new rules for their games so all twenty numbers could play. Let's see if you can discover the rule for their calculations.

Write the calculations on the board.

$$5 * 3 = 8$$

$$14 * 11 = 5$$

$$7 * 5 =$$

T: Can you guess which number 7 * 5 is?

If students respond "2", remind them that this is a game for the twenty numbers from 0 to 19. When the correct response "12" is given, write it on the board. Encourage students to be thinking about the rule, but not to give it away when they know. Continue in this way for the following calculations (correct responses are in the boxes) until it appears that many students understand the rule. Then ask a student to describe the rule.

$$19 * 10 = \boxed{9}$$

$$18 * 12 = \boxed{10}$$

$$13 * 3 = \boxed{16}$$

$$17 * 16 = \boxed{13}$$

$$4 * 16 = \boxed{0}$$

9 L

T: Who can explain the rule?

S: Add the numbers. If the sum is less than 20, keep it. If it is 20 or more, subtract 20 and keep what's left.

Other explanations may be proposed; for each one let the class check it by applying it to the above calculations. When the rule has been discovered and adequately explained, introduce the new symbol "$+_{20}$". (Read "$+_{20}$" as "plus with twenty friends".)

T: Now we know the star is for addition with twenty friends. To help us remember the rule, let's use "$+_{20}$" instead of "$*$" for this game.

Replace "$*$" by "$+_{20}$" in all the calculations on the board. Continue with some other problems to give all the students a chance to calculate with $+_{20}$. The following examples have correct student responses in the boxes.

$8 +_{20} 13 = \boxed{1}$ $\qquad$ $9 +_{20} 8 +_{20} 5 +_{20} 10 = \boxed{12}$

$10 +_{20} 15 = \boxed{5}$ $\qquad$ $13 +_{20} \boxed{16} = 9$

$19 +_{20} 19 = \boxed{18}$ $\qquad$ $\boxed{17} +_{20} 6 = 3$

T: For the next problem, you must put the same number in both boxes.

Write this problem on the board.

$$\square +_{20} \square = 6$$

T: Which number could be in the boxes?

S: 3, because 3 + 3 = 6.

S: 13, because 13 + 13 = 26 and then subtract 20 to get 6.

If students suggest 23, remind them that only the twenty numbers 0 to 19 are in the game.

Continue in the same way with this problem.

$\square +_{20} \square = 16$ (Answers: 8 and 18)

T: For the next problem, you may put the same number in the triangle and the box, but you may also put different numbers.

Write this problem on the board.

$\triangle +_{20} \square = 12$

Let students make suggestions and list correct solutions on the board. There are twenty possibilities (pairs of numbers), but students should quickly notice that you can always reverse the order of a pair and get another solution. So we need only consider eleven solutions.

Solutions: $\triangle +_{20} \square = 12$

0	12
1	11
2	10
3	9
4	8
5	7
6	6
13	19
14	18
15	17
16	16

9 L

If this problem is solved quickly, continue with another similar problem.

Solutions: $\triangle +_{20} \square = 5$

0	5
1	4
2	3
6	19
7	18
8	17
9	16
10	15
11	14
12	13

T: The twenty numbers 0 to 19 also play a multiplication game. We use "$\times_{20}$" (write the symbol on the board and read: times with twenty friends) for this game. Let's see if you know the rule.

Let the students make several calculations (responses are in the boxes) until it appears that many students understand this rule. Then ask a student to explain the rule.

$5 \times_{20} 3 = \boxed{15}$

$5 \times_{20} 5 = \boxed{5}$

$6 \times_{20} 6 = \boxed{16}$

$10 \times_{20} 2 = \boxed{0}$

$4 \times_{20} 12 = \boxed{8}$

$7 \times_{20} 9 = \boxed{3}$

$5 \times_{20} 11 = \boxed{15}$

T: Who can explain the rule?

S: Multiply the numbers. If the product is less than 20, keep it. If it is 20 or more, subtract 20's until you get a number less than 20.

Let the class check the rule with some other problems.

$9 \times_{20} 8 = \boxed{12}$

$10 \times_{20} 15 = \boxed{10}$

$\square \times_{20} 6 = 4$ (Answers: 4 and 14)

$10 \times_{20} \square = 0$ (Answers: 0, 2, 4, 6, 8, 10, 12, 14, 16, and 18)

Exercise 2

T: First we had ten numbers playing the DANCING FRIENDS games. Then we invited ten more numbers and there were twenty friends. Suppose we invite twenty more numbers so that there are forty friends to play some games. Which forty numbers do you think we should invite?

9 L

S: The whole numbers 0 to 39.

T: Very good. Now, let's see if you can discover the rules for addition and multiplication with forty friends. (Write "$+_{40}$" and "$\times_{40}$" on the board.)

Write these problems on the board; ask the students to solve them and to explain their answers.

$$25 +_{40} 26 = \square$$

S: 11, because 25 + 26 = 51 and subtract 40 leaves 11.

$$36 +_{40} 24 = \square$$

S: 20, because 36 + 24 = 60 and subtract 40 leaves 20.

$$9 \times_{40} 8 = \square$$

S: 32, because 9 x 8 = 72 and subtract 40 leaves 32.

$$8 \times_{40} 11 = \square$$

S: 8, because 8 x 11 = 88 and subtract two 40's (that is, 80) leaves 8.

T: Very good. You must subtract 40's until you get a number less than 40 because only the numbers 0 to 39 are in the game.

Continue with some other problems until the class appears to have a good idea about addition and multiplication with forty friends.

$13 +_{40} 16 +_{40} 17 = \boxed{6}$ $\qquad$ $6 \times_{40} 15 = \boxed{10}$

$17 +_{40} \boxed{33} = 10$ $\qquad$ $5 \times_{40} 25 = \boxed{5}$

$\boxed{12} +_{40} 28 = 0$ $\qquad$ $10 \times_{40} \square = 0$

(Solutions: 0,4,8,12, 16,20,24,28,32,36)

T: Suppose we invite forty more numbers to play in some games. Which eighty numbers do you think we should include?

S: The whole numbers 0 to 79.

T: Very good. Let's see if you know how these numbers play the games, addition and multiplication with eighty friends.

Write these problems on the board; ask students to solve them and to explain their answers.

$37 +_{80} 26 = \boxed{63}$ $\qquad$ $7 \times_{80} 7 = \boxed{49}$

$61 +_{80} 45 = \boxed{26}$ $\qquad$ $9 \times_{80} 9 = \boxed{1}$

$57 +_{80} 23 = \boxed{0}$ $\qquad$ $9 \times_{80} 11 = \boxed{19}$

$15 +_{80} 25 +_{80} 55 = \boxed{15}$ $\qquad$ $10 \times_{80} 20 = \boxed{40}$

9 L

The explanations should suggest that in each problem you add (or multiply) the numbers in the usual way and if the result is less than 80, keep it. If the result is 80 or more, subtract 80's (or a multiple of 80) until you get a number less than 80.

Exercise 3

One at a time, ask the class to do these calculations.

$$2 \times_{80} 2 = \boxed{4}$$

$$2 \times_{80} 2 \times_{80} 2 = \boxed{8}$$

$$2 \times_{80} 2 \times_{80} 2 \times_{80} 2 = \boxed{16}$$

$$2 \times_{80} 2 \times_{80} 2 \times_{80} 2 \times_{80} 2 = \boxed{32}$$

T: It's tiring to write "$2\times_{80}$" so many times, so we will use a shorthand notation.

Write the appropriate power of 2 next to each calculation on the board and continue with more calculations in the same pattern.

$$2 \times_{80} 2 = 2^2 = 4$$

$$2 \times_{80} 2 \times_{80} 2 = 2^3 = 8$$

$$2 \times_{80} 2 \times_{80} 2 \times_{80} 2 = 2^4 = 16$$

$$2 \times_{80} 2 \times_{80} 2 \times_{80} 2 \times_{80} 2 = 2^5 = 32$$

$$2^6 = \boxed{64}$$

$$2^7 = \boxed{48}$$

You may need to get an explanation for this last calculation before continuing; for example, $2^6 = 64$ so $2^7 = 2 \times_{80} 64 = 48$ because $2 \times 64 = 128$ and $128 - 80 = 48$. Continue the calculations and try to arrange your board to suggest the emerging pattern.

$2^4 = 16$	$2^8 = \boxed{16}$	$2^{12} = \boxed{16}$
$2^5 = 32$	$2^9 = \boxed{32}$	$2^{13} = \boxed{32}$
$2^6 = 64$	$2^{10} = \boxed{64}$	$2^{14} = \boxed{64}$
$2^7 = 48$	$2^{11} = \boxed{48}$	$2^{15} = \boxed{48}$

T: Do you see a pattern in these calculations?

S: The solutions keep repeating 16, 32, 64, 48; 16, 32, 64, 48; and so on. These four numbers are always the answers.

S: Yes, but not until we were past 2^3.

T: Can you predict what 2^{25} will be?

S: 32, because $2^{16} = 16$ and $2^{20} = 16$ and $2^{24} = 16$, so $2^{25} = 32$.

T: Can you predict what 2^{44} will be?

S: 16, because 44 is a multiple of 4 and whenever there is a multiple of 4 for the power, we get 16.

Do not expect such precise explanations immediately. Let the students experiment with describing the pattern for a while. If the class seems to be

confident about the pattern, ask them to try to predict the following calculations. Otherwise, proceed to individual work on Worksheets L14 and L15. You could still challenge some of your better students with these problems.

$2^{163} = \square$ (Answer: 48)

$2^{1,977} = \square$ (Answer: 32)

Name _____ L 14 *

Complete:

59 + 36 = 15

34 + 46 + 62 = 62

47 + 26 + 72 = 65

10 × 42 = 20

3 × 40 × 5 = 40

4 × 40 × 78 = 0

The number in the box could be:

□ + 52 = 15 → 43

4 × □ = 40 → 10 or 30 or 50 or 70

□ × 5 = 60 → 12 or 28 or 44 or 60 or 76

Name _____ L 15 **

Complete:

$3^1 = 3$

$3^2 = 3 \times 3 = 9$

$3^3 = 3 \times 3 \times 3 = 27$

$3^4 = 3 \times 3 \times 3 \times 3 = 1$

$3^5 = 3$

$3^6 = 9$

$3^7 = 27$

$3^8 = 1$

$3^9 = 3$

If you see a pattern, find

$3^{27} = 27$

(Continue on the back.)

Complete:

$5^1 = 5$

$5^2 = 5 \times 5 = 25$

$5^3 = 5 \times 5 \times 5 = 45$

$5^4 = 5 \times 5 \times 5 \times 5 = 65$

$5^5 = 5$

$5^6 = 25$

$5^7 = 45$

$5^8 = 65$

$5^9 = 5$

If you see a pattern, find

$5^{27} = 5$

11L

1.7 The Kite with Four Tails #1

Distribute several sheets of paper and colored pencils to each student.

Exercise 1

T: Do you remember the addition and multiplication games with ten friends, and with twenty friends, and with forty friends, and with eighty friends? Let's try some calculations.

Write the problems on the board and elicit student responses as given in the boxes. Ask the students to write their answers on their papers, and then check collectively with explanations.

$8 +_{10} 9 = \boxed{7}$ $\quad 7 \times_{10} 9 = \boxed{3}$ $\quad \boxed{6} +_{10} 6 = 2$

$3 \times_{10} \boxed{7} = 1$ $\quad (3 \times_{10} \boxed{9}) +_{10} 5 = 2$

T: How many friends?

S: Ten, the whole numbers from 0 to 9.

Continue with these problems.

$18 +_{20} 15 = \boxed{13}$ $\quad 7 \times_{20} 11 = \boxed{17}$ $\quad \boxed{7} +_{20} 16 = 3$

$3 \times_{20} \boxed{10} = 10$ $\quad (3 \times_{20} \boxed{5}) +_{20} 7 = 2$

$\square +_{20} \square = 8$ (Answers: 4 and 14)

For this last problem, you should remind the class that the same number goes in each box.

T: How many friends?

S: Twerty, the whole numbers from 0 to 19.

Continue in the same manner with calculations involving various numbers of friends.

$24 +_{40} 17 = \boxed{1}$ $\quad 24 +_{30} 17 = \boxed{11}$ $\quad 42 +_{50} 37 = \boxed{29}$

$8 \times_{40} 7 = \boxed{16}$ $\quad 5 \times_{30} 9 = \boxed{15}$ $\quad 10 \times_{50} 12 = \boxed{20}$

$75 +_{100} 48 = 23$ $\quad 28 \times_{100} 10 = \boxed{80}$ $\quad 12 \times_{100} 12 = \boxed{44}$

$55 +_{80} \boxed{39} = 14$ $\quad 55 \times_{80} 3 = \boxed{5}$ $\quad 10 \times_{80} \square = 0$

(Answers: 8, 16, 24, 32, 38, 56, 64, 72)

Erase the board before going on to Exercise 2.

Exercise 2

T: Now we will use just the games with eighty friends, the whole numbers from 0 to 79.

Write this sequence (on the following page) of calculations on the board. Let the students provide the numbers in the boxes. Try to arrange the board so that a pattern is easily observed by the class.

$2^1 = 2$

$2^2 = 2 \times_{80} 2 = \boxed{4}$

$2^3 = 2 \times_{80} 2 \times_{80} 2 = \boxed{8}$

$2^4 = \boxed{16}$	$2^8 = \boxed{16}$	$2^{12} = \boxed{16}$
$2^5 = \boxed{32}$	$2^9 = \boxed{32}$	$2^{13} = \boxed{32}$
$2^6 = \boxed{64}$	$2^{10} = \boxed{64}$	$2^{14} = \boxed{64}$
$2^7 = \boxed{48}$	$2^{11} = \boxed{48}$	$2^{15} = \boxed{48}$

T: Do you see a pattern here?

S: The numbers 16, 32, 64, and 48 keep repeating.

S: Every time it is 2 with a multiple of 4 power, then the number is 16.

NOTE: You might not receive such clearly stated explanations of the pattern; however, your students will probably express the same idea. Encourage them to note that when the power is a multiple of 4, then the number 2 to that power is 16.

T: Yes. Let's see if you can use this pattern to solve these problems.

Write the problems on the board and ask the students to do them on their papers. Answers are in the boxes.

$2^{20} = \boxed{16}$

$2^{23} = \boxed{48}$

$2^{41} = \boxed{32}$

$2^{45} = \boxed{32}$

$2^{50} = \boxed{64}$

$2^{108} = \boxed{16}$

$2^{110} = \boxed{64}$

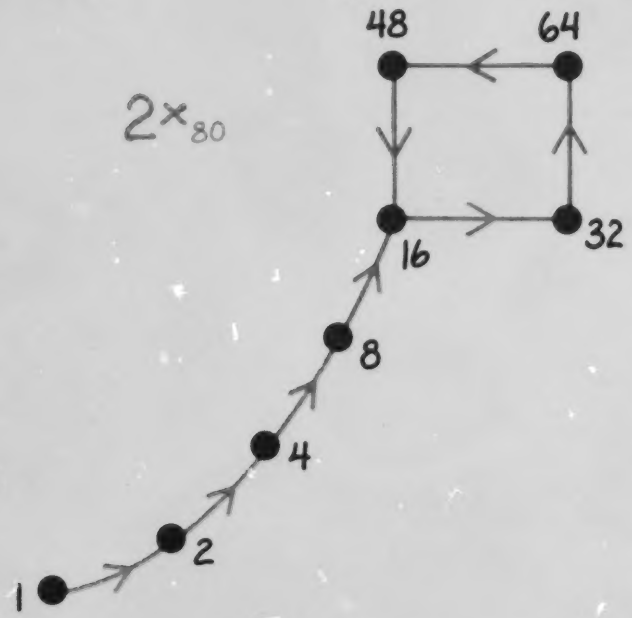
$2^{403} = \boxed{48}$

Exercise 3

T: Let's build a road starting at 1 and using blue arrows for $2 \times_{80}$ (read: two times with eighty friends).

11 L

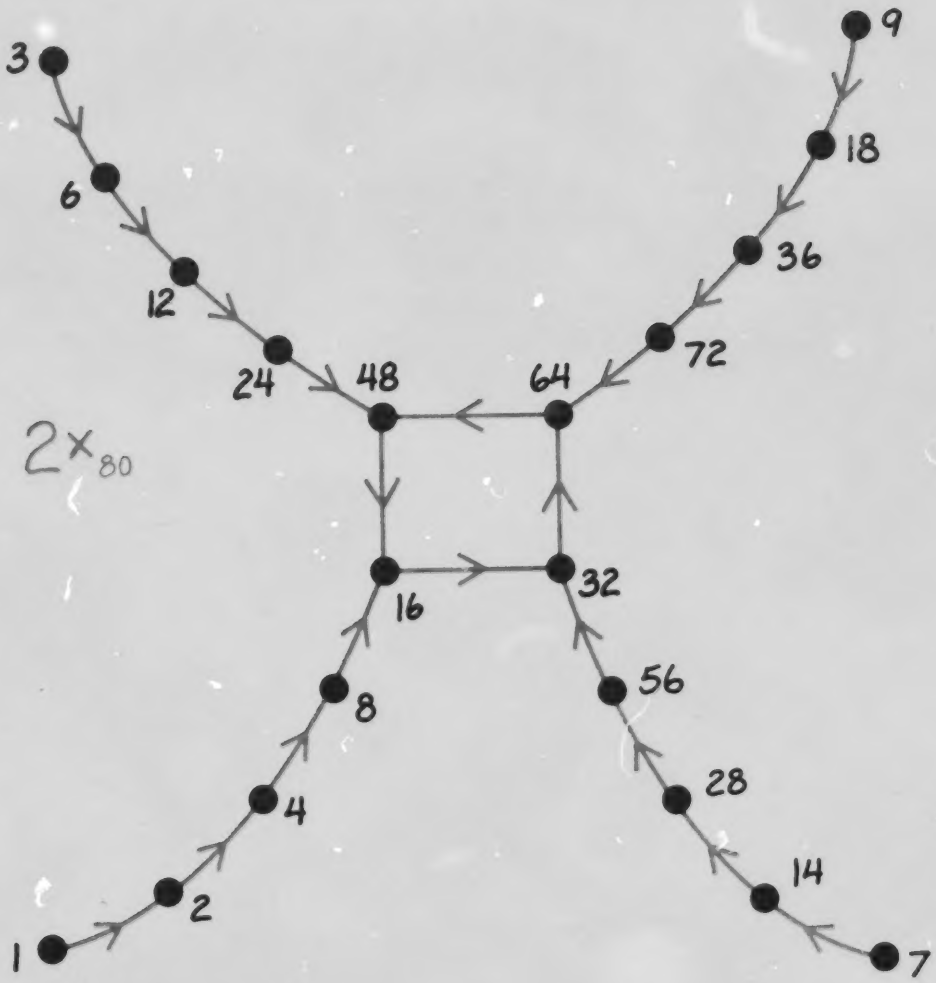
Draw the following picture with the help and active participation of the class. Encourage the class to comment on the familiarity of the picture.



S: It's the kite again.

Divide the class into three groups. Tell the students that they will draw on their papers a road using blue arrows for $2 \times_{80}$. Ask the students in the first group to start their roads at 3, the second group to start at 7, and the third group to start at 9. Allow several minutes for this individual work.

Choose one member from the first group to draw the road starting at 3 on the board and to show how it is connected to the road starting at 1 already on the board. Then choose a member from the second group to do the same for the road starting at 7; and then a member from the third group to do the same for the road starting at 9. You can influence how the resulting picture will appear by putting the dots for 3, 7, and 9 on the board yourself. Each of the new roads is a kite which connects to the first kite at the center, and the resulting picture looks like a four-tailed kite as in the next illustration. Encourage the class to comment on the picture.



S: Now the kite has four tails.

Perhaps some students will recognize that the repeating numbers from the calculations in Exercise 2 will appear in the center of the picture. If so, let them try to explain but do not force the idea.

12 L

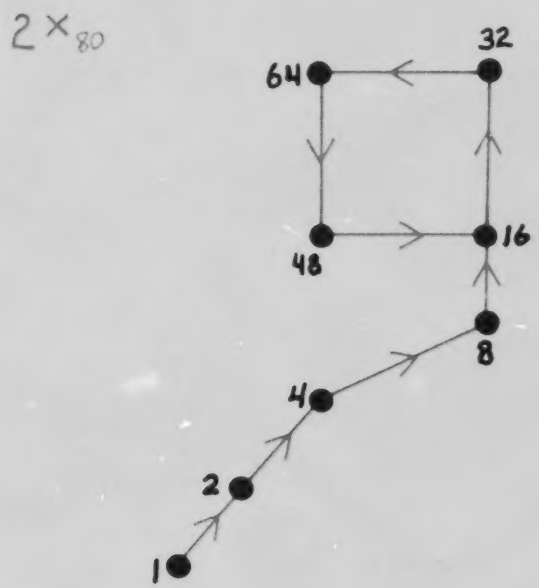
1.8 The Kite with Four Tails #2

If you think it is necessary, begin the lesson by reviewing briefly addition and multiplication with eighty friends.

Exercise 1

T: Do you remember the kite with four tails we got when we built roads using $2 \times_{80}$ arrows? We had roads starting at 1, at 3, at 7, and at 9 which were connected at the center.

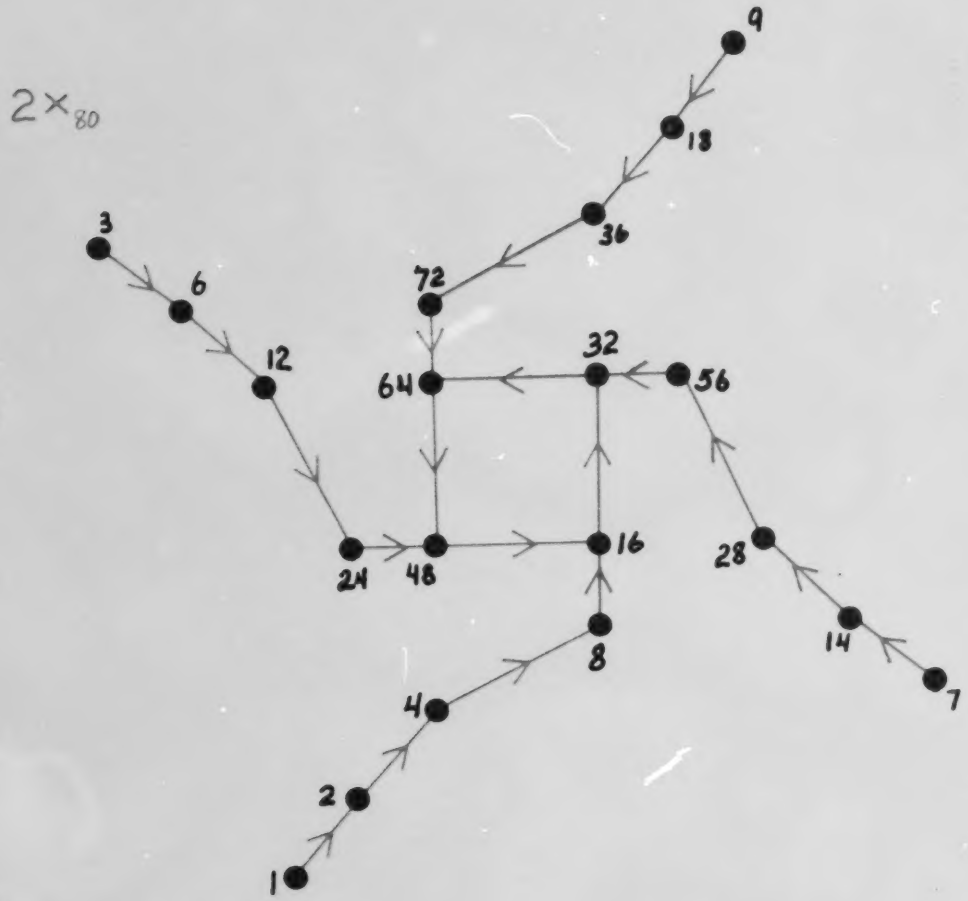
With class participation, draw the $2 \times_{80}$ road starting at 1 and layout the picture on the board so that it resembles a kite. Be sure to center the picture on the board so that the other three tails can be added.



T: Do you remember that we found three more tails for this kite? Which road connected to the kite at 64 and gave us another tail at that corner?

S: The one starting at 9.

Draw the $2\times_{80}$ road starting at 9 connecting to the kite at 64. Then repeat the question for the road which connects to the kite at 48 (the one starting at 3) and the road which connects to the kite at 32 (the one starting at 7). In this way, you should reproduce the kite with four tails as in the next illustration.

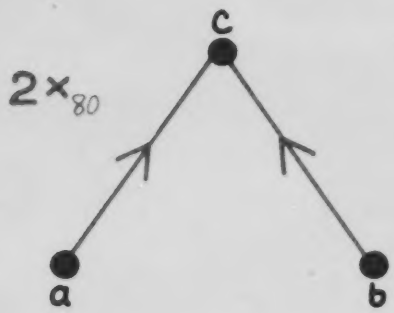


Do <u>not</u> erase the four-tailed kite picture before going on to the next exercise.

12 L

Exercise 2

On another part of the board, draw this arrow picture and table.



a	b	c

T: Which numbers could be at a, b, and c? Remember, eighty friends play so only the whole numbers 0 to 79 can be in this picture.

S: 10 at a, 50 at b, and 20 at c.

S: 8 at a, 48 at b, and 16 at c.

Let the students find many possibilities, check each suggestion collectively, and enter correct choices for a, b, and c in the table. Students usually observe that they can reverse a and b to get another solution. If so, you could begin to enter both solutions at one time. After a while your table might look like the one in the next illustration.

a	b	c
10	50	20
8	48	16
2	42	4
0	0	0
46	6	12
6	46	12
32	72	64
72	32	64
48	8	16
50	10	20

Encourage students to comment on any patterns they see in the table.

T: Suppose 14 were at a. Which numbers would be at c and b?

S: 28 would be at c because $2 \times_{80} 14 = 28$, and 54 would be at b because because $2 \times_{80} 54 = 28$.

T: Suppose 64 were at a. Which numbers would be at c and b?

S: 48 would be at c and 24 would be at b.

T: Suppose 36 were at c. Which numbers could be at a and b?

S: a could be 18 and b could be 58, or the reverse.

T: Do you see any patterns in this table?

S: b is always 40 more or 40 less than a.

12 L

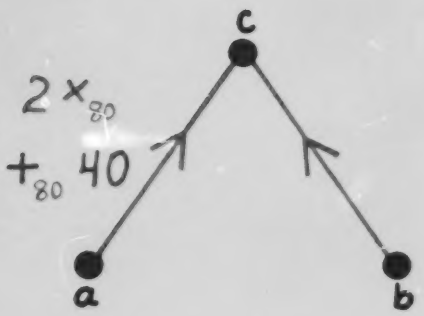
T: Is there any eighty friends games that could relate a and b?

S: $a +_{80} 40 = b$.

S: $b +_{80} 40 = a$.

Write "$+_{80}40$" in red on the board.

T: Very good. How can I show those relations in my picture?

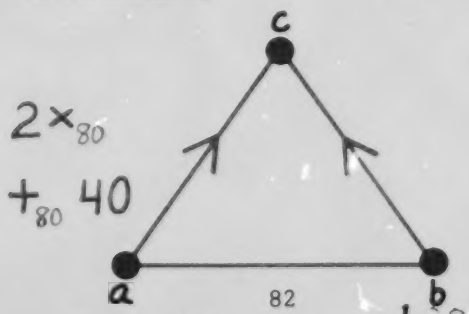


a	b	c
10	50	20
8	48	16
2	42	4
0	0	0
46	6	12
6	46	12
32	72	64
72	32	64
48	8	16
50	10	20

S: Draw red arrows from a to b and from b to a.

S: Since we have red arrows in both directions, we could just use a red cord.

Draw a red cord between a and b.



Direct the class' attention back to the four-tailed kite.

T: Where can we draw red cords in this kite picture?

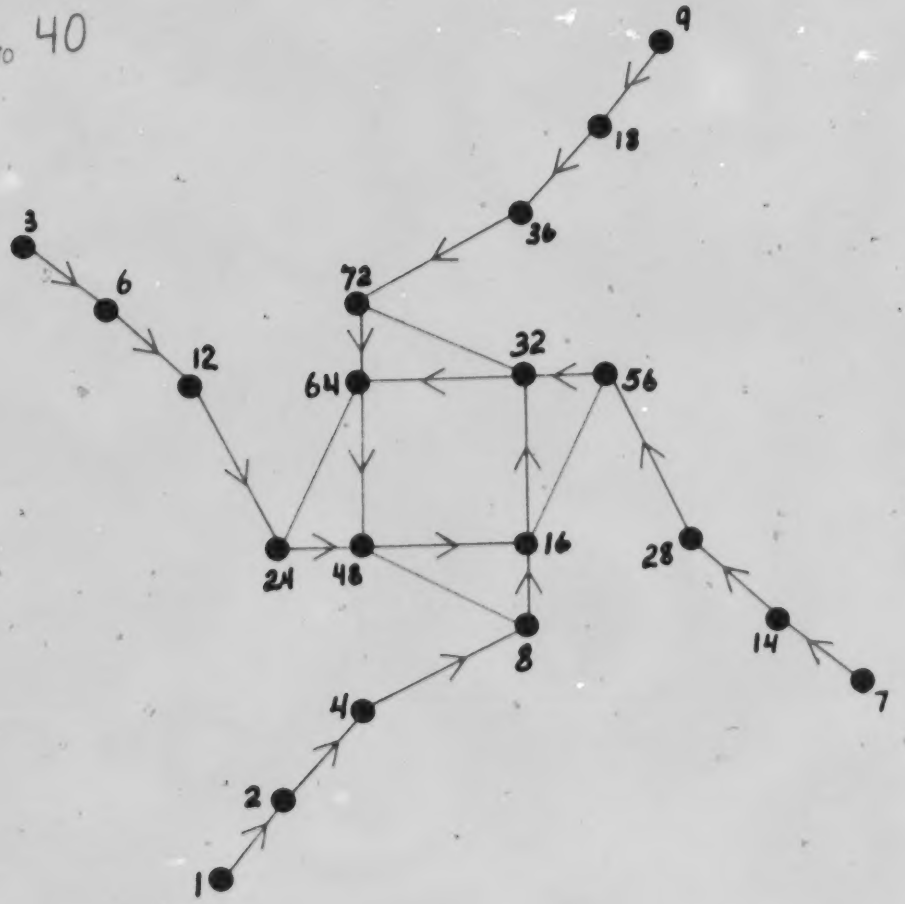
S: Between 8 and 48.

S: Between 16 and 56.

Draw the cords as they are suggested. The class should quickly find four red cords.

$2 \times_{80}$

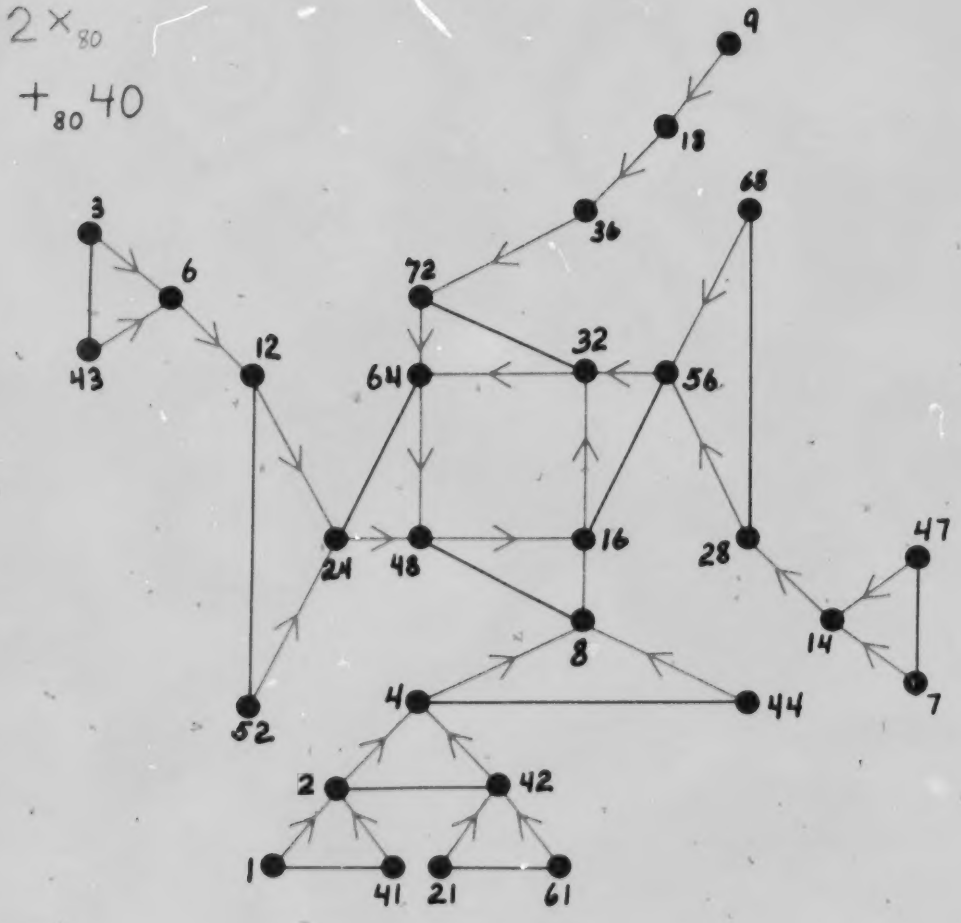
$+_{80}$ 40



1.29

12 L

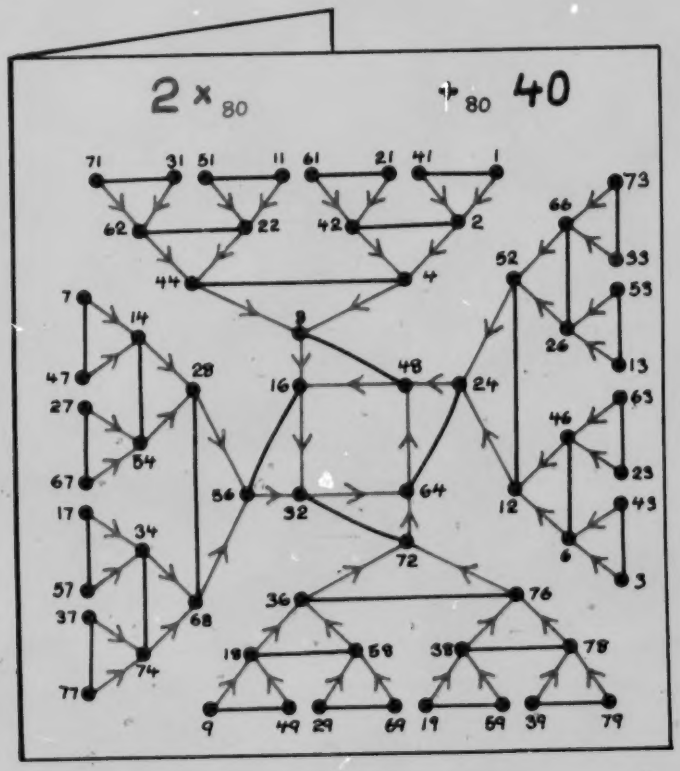
Then, suggest that the students put more of the numbers (0 to 79) in the picture and try to find other places to draw red cords. Continue this activity until several more red cords have been located. For example, your picture might look like this.



110

T: Do you think all the whole numbers from 0 to 79 can be attached to this picture?

You will likely have varied responses to this question. Let the class discuss the situation for a moment, and then display the Language Poster #2 for IG III with the left piece folded back so as not to be visable.



T: This picture shows all the $2x_{80}$ roads that will attach to our kite picture.

Let the class examine the poster for a minute or two. The students may first need to observe that indeed the picture is like the one on the board with more $2x_{80}$ roads attached.

T: Are all the whole numbers from 0 to 79 in this picture?

S: No, 0 is missing and so is 40.

12 L

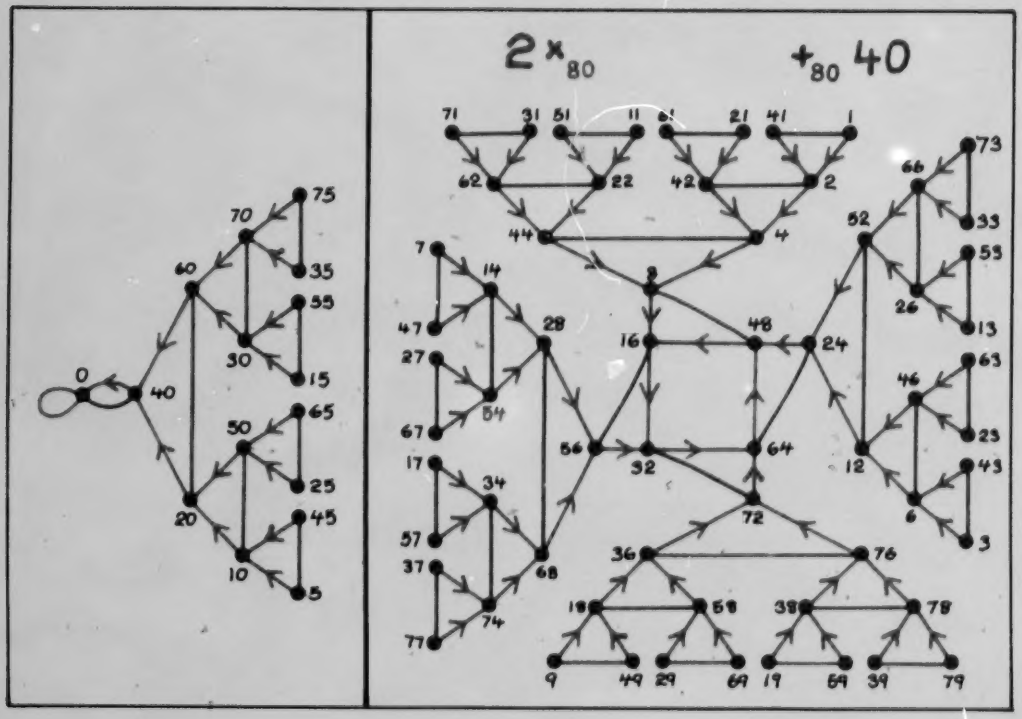
S: There are only sixty-four numbers in this picture, not eighty.

S: All the multiples of 10 are missing.

S: All the multiples of 5 are missing.

Turn out the folded back portion of the poster as you say:

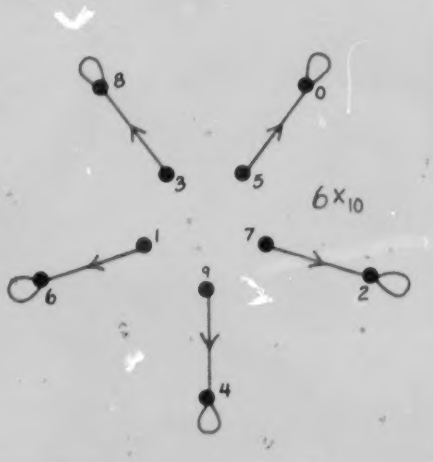
T: You are right, and here are all the multiples of 5 in their own $2\times_{80}$ picture.



Let the class comment on the picture for a couple minutes. Worksheets L16 and L17 are available for individual work.

Name _____ L 16 *

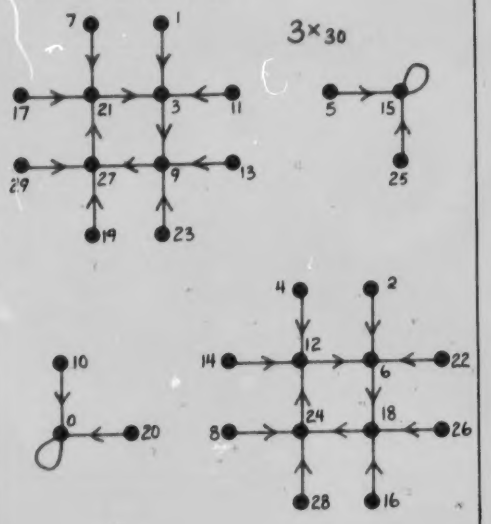
Put the numbers 0, 1, 2, 3, 4, 5, 6, 7, and 8 on this picture



(Several arrangements of the numbers in this picture are correct. It is only necessary that 1 and 6, 3 and 8, 5 and 0, 7 and 2, 9 and 4 be connected.)

Name _____ L17 **

Put the numbers from 0 to 29 on this picture



(The numbers 5, 15, and 25 could be interchanged with 10, 0 and 20.)

2

STRING GAMES and LOGICAL THINKING

The language of strings is used as a vehicle to present various situations involving sets which stimulate logical thinking. The situations presented in the lessons of this chapter are similar to those experienced earlier in the CSMP program, and special attention is given to situations involving the already very familiar String Games. Continuing the experiences in the L strand of CSMP Mathematics for the Intermediate Grades, Part II, the String Game situations involve the students in thinking critically about the relations "is a divisor of", "is a multiple of", "is larger than", etc., as well as challenge them to think logically about the relationships between various sets and to give persuasive arguments. The situations are rich, they encourage deductive reasoning on the part of students, and with the aid of string pictures, they draw upon the students' numerical knowledge to solve problems.

CAPSULE LESSON SUMMARIES

Section 2.1: Analyze the information obtained from the starting clues of a String Game. Using this information, determine whether a given statement is true, false, or you can't tell. Play the String Game.
(Worksheets L18, L19, L20, L21)

Section 2.2: Put numbers in a string picture where one string is determined and there are two possibilities for the other string. Discuss in which regions numbers that cannot be placed could be. In String Game situations, analyze the information given by hatching clues. Play the String Game.

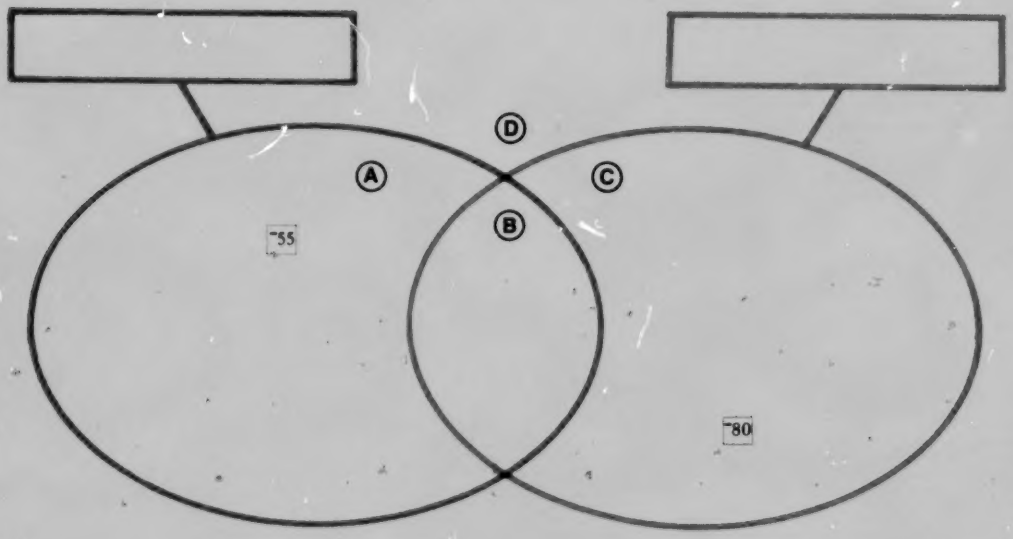
Section 2.3: Continue exercises similar to the first one in Section 2.2, and extend the problem to include a situation where there are two possibilities for each of the two strings.
(Worksheets L22, L23, L24, L25)

Section 2.4: Continue with exercises similar to those of Section 2.1. Extend the situation to include hatching in the string picture as an additional clue. Play the String Game.
(Worksheets L26, L27, L28, L29, L30)

2 L

2.1 String Game Analysis #1

Before the lesson begins, set up your board for the String Game as illustrated below. Tape two demonstration analysis sheets (included in the Numerical String Game Kit) on the board; one to the left of the red string and one to the right of the blue string. The four regions in this picture have been labeled "A", "B", "C", and "D" to make it easier to discuss them.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN 10
SMALLER THAN 50	SMALLER THAN 10	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN 10
SMALLER THAN 50	SMALLER THAN 10	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: This is the starting situation for a String Game. Here is the list of possibilities for the red string (point to the analysis sheet at the left) and here is the list of possibilities for the blue string (point to the analysis sheet at the right). There are two clues; two numbers have already been placed. What information do these clues give us about the strings?

S: The red string cannot be for the multiples of 2, because -55 is not a multiple of 2.

S: The red string cannot be for the multiples of 5, because -80 is a multiple of 5 and is outside the red string.

Use a crayon or heavy felt-tip pen to cross off a label from the appropriate list as it is mentioned. Insist upon a clear explanation for why a string cannot have the label before you cross it off. Sometimes a student may suggest crossing off several labels at one time with one explanation; for example,

S: The red string (and also the blue string) cannot have any of the "positive divisor of" labels because there is a negative number inside the string.

Let the students eliminate as many labels as they can from both the RED and BLUE lists. When all the information from these clues has been discussed, your class should find that the red string is determined (it is for the odd numbers) and that there are still three possibilities for the blue string (multiples of 2 or multiples of 4 or multiples of 10).

RED

~~MULTIPLES OF 2~~	~~MULTIPLES OF 8~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	ODD NUMBERS	~~POSITIVE PRIME NUMBERS~~

BLUE

MULTIPLES OF 2	~~MULTIPLES OF 8~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 59~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: Now, I will give you some statements. With this information, you must say whether the statement is true or false or you can't tell. Be ready to explain your answer. I have labeled the regions "A", "B", "C", and "D" to make it easier to talk about them.

2 L

Statements you could use with this situation are on the left and correct student responses are on the right. Encourage your students to check each others responses and to discuss them if necessary.

T: 105 is in the red string.

S: True, because 105 is an odd number.

T: 105 is in region B.

S: False. 105 cannot be in the blue string because 105 is not a multiple of 2, not a multiple of 4, and not a multiple of 10.

T: So, where must we put 105 ?

S: In region A.

Put 105 in region A of the string picture.

T: 50 is in the red string.

S: False, because 50 is not an odd number.

T: 50 is in the blue string.

S: Can't tell. It would be in the blue blue string if the blue string were for multiples of 2 or for multiples of 10. It would not be in the blue string if the blue string were for multiples of 4.

T: So, 50 could be in which regions ?

S: Region C or region D.

T: 45 is in region D.

S: False, 45 is an odd number so it must be in the red string.

T: 45 is inside both strings.

S: False. 45 is not in the blue string because 45 is not a multiple of 2, not a multiple of 4, and not a multiple of 10.

T: So where must we put 45? S: In region A.

Put 45 in region A of the string picture.

T: 40 is inside both strings. S: False. 40 is not an odd number.

T: 40 is outside both strings. S: False. 40 is in the blue string because it is a multiple of 2, and of 4, and of 10.

T: So, where must we put 40? S: In region C.

Put 40 in region C of the string picture.

T: Even though we do not know for sure what the blue string is for, there is one region which can be hatched. Do you see which one?

S: Region B, because multiples of 2, of 4, and of 10 are all even numbers.

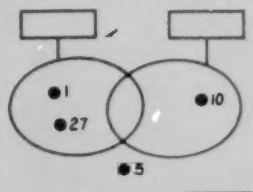
Hatch region B in your picture.

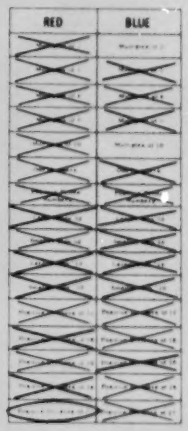
Distribute Worksheets L18 and L19. Let the students work independently on these problems. Students who finish these worksheets quickly should be given Worksheets L20 and L21. After about 15 minutes of independent work, collectively discuss the answers for L18 and L19. This discussion should be similar to that described for the situation above.

For the rest of the lesson, play the String Game. See the appendix to this chapter: The String Game, for a description of the different versions of the game and many examples of games you could play.

Name _____ **L 18** *

Cross out the labels that the strings cannot have. Some are done for you.





The label for the red string is [positive divisors of 27]

The label for the blue string could be

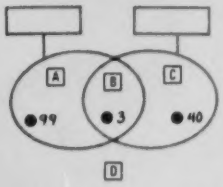
[multiples of 1] or [multiples of 10]

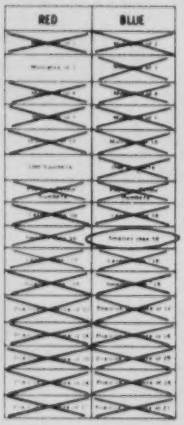
Circle one of the following: T (True) F (False) CT (Can't Tell)

1) 9 is in the red string. (T) F CT
2) 7 is in the blue string. T (F) CT
3) 99 is outside both strings. (T) F CT
4) 2 is in the blue string. T F (F)
5) 3 is in both strings. T (T) CT
6) 4 is outside both strings. T F (F)

Name _____ **L 19** **

Cross out the labels the strings cannot have. Some are done for you.





The label for the blue string is [smaller than 50]

The label for the red string could be

[multiples of 3] or [odd numbers]

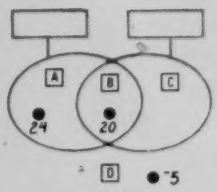
In the string picture, four regions are labeled: A, B, C, D.

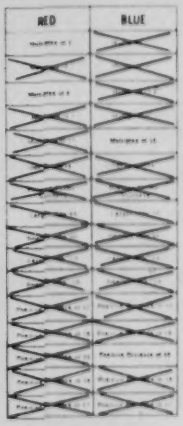
Circle one of the following: T (True) F (False) CT (Can't Tell)

1) 20 is in region C. (T) F CT
2) 2 is in region B. T (F) CT
3) 105 is in region B. T (F) CT
4) 105 is in region A. (T) F CT
5) 7 is in region B. T F (F)
6) 7 is in region D. T (F) CT
7) 100 is in region D. (T) F CT
8) 60 is in region A. T F (F)

Name _____ **L 20** ***

Cross out the labels that the strings cannot have.





The label for the red string could be

[multiples of 2] or [multiples of 4]

The label for the blue string could be

[multiples of 10] or [positive divisors of 20]

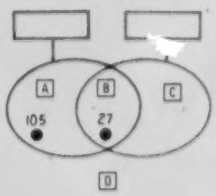
In the string picture, four regions are labeled: A, B, C, D.

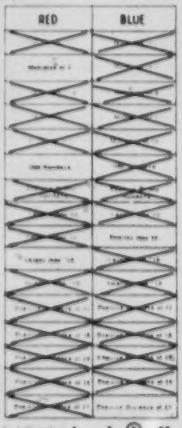
Circle one of the following: T (True) F (False) CT (Can't Tell)

1) 4 is in region C. T (F) CT
2) 2 is in region D. T F (F)
3) 10 is in region A. T (F) CT
4) 8 is in region A. (T) F CT
5) 10 is in region C. T F (F)
6) 100 is in region B. T F (F)
7) 0 is in region D. T (F) CT
8) 1 is in region C. T F (F)
9) -1 is in region D. (T) F CT
10) 50 is in region C. T F (F)

Name _____ **L 21** ****

Cross out the labels that the strings cannot have.





In this string picture, four regions are labeled: A, B, C, D.

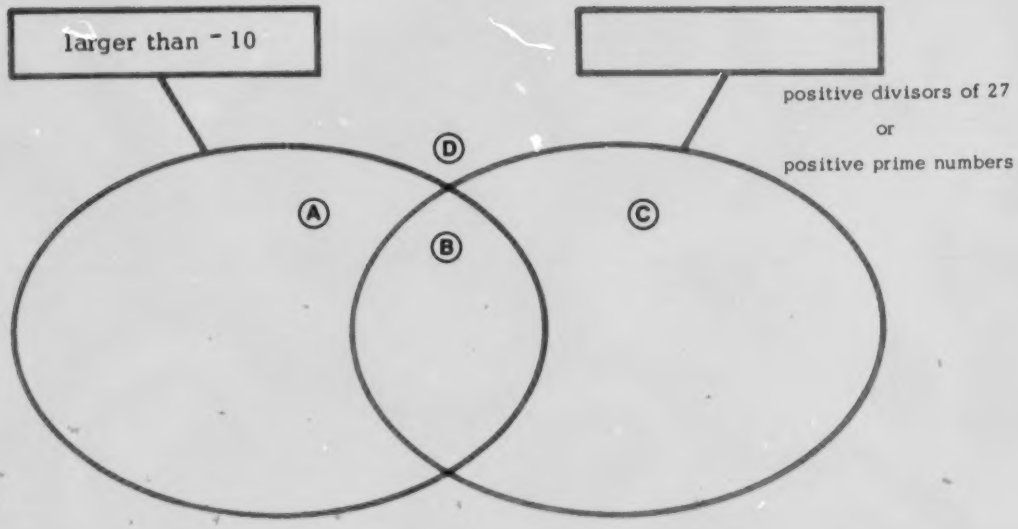
Circle one of the following: T (True) F (False) CT (Can't Tell)

1) 45 is in region C. T (T) CT
2) 45 is in region A. T F (F)
3) -15 is in region B. T F (F)
4) 9 is in region B. (T) F CT
5) 3 is in region C. T (T) CT
6) 1 is in region A. T (T) CT
7) -80 is in region D. T F (F)
8) -100 is in region A. T (T) CT
9) 99 is in region A. (T) F CT
10) 60 is in region D. T F (F)

2.2 String Game Analysis #2

Exercise 1

Draw this string picture on the board.



T: Imagine that we are playing the String Game. We determined that the red string is for the numbers larger than -10, and there are still two possibilities for the blue string. It could be for the positive divisors of 27 or for the positive prime numbers.

Write these labels on the board near the blue string, but not in the blue box.

T: I am going to give you six numbers from the String Game. You have to place these numbers in the correct region, if possible. If it is not possible, tell which regions the number could be in.

Aside from the picture, write this list of numbers on the board.

3, 4, 0, 1, -80, 7

5 L

Let students respond, in any order they wish, that a number is in a certain region or there are two regions that it could be in. Encourage the class to check each response before continuing. Put the numbers that can be placed for sure in a region, in your picture. Write "CT" for "Can't Tell" next to numbers which cannot be placed for sure. Correct responses are given below.

S: $^{-}80$ is in region D, because $^{-}80$ is <u>not</u> larger than $^{-}10$ and $^{-}80$ is <u>not</u> a positive number (hence, not a positive divisor of 27 and not a positive prime number).

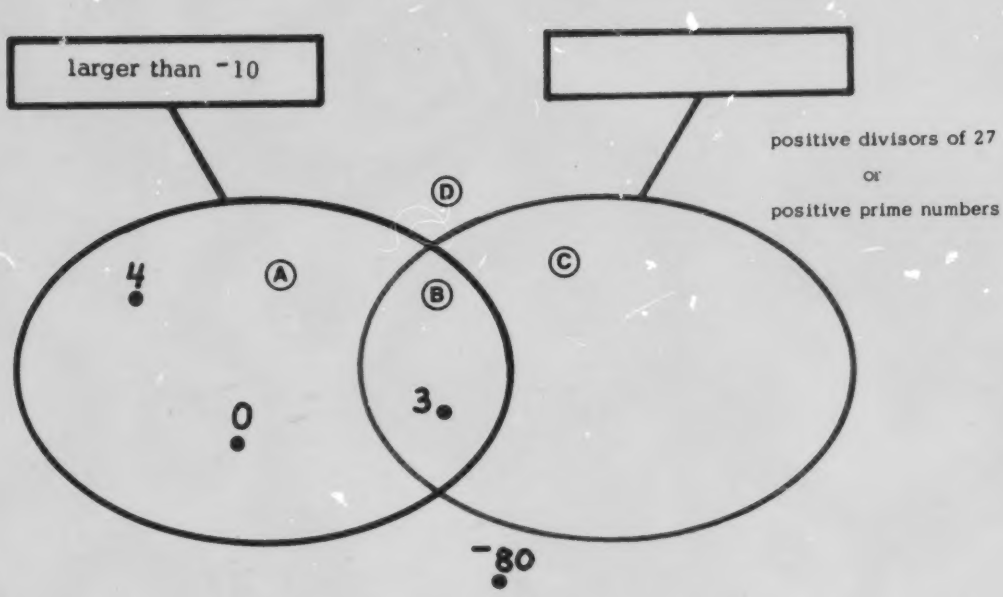
S: 0 is in region A, because 0 is larger than $^{-}10$ but is <u>not</u> a positive number.

S: 1 must be in the red string because it is larger than $^{-}10$, so 1 could be in region A or in region B. It is in region A if the blue string is for the positive prime numbers (1 is not a positive prime number). It is in region B if the blue string is for the positive divisors of 27 (1 is a positive divisor of 27).

S: 3 is in region B, because 3 is larger than $^{-}10$, it is a positive divisor of 27, and it is a positive prime number.

S: 4 is in region A, because 4 is larger than $^{-}10$, but it is not a positive divisor of 27 and it is not a positive prime number.

S: 7 must be in the red string because it is larger than $^{-}10$, so 7 could be in region A or B. It is in region A if the blue string is for positive divisors of 27, and it is in region B if the blue string is for positive prime numbers.



CT 1, CT 7

T: Even though we still do not know what the blue string is for, one of the regions A, B, C, or D can be hatched. Do you see which one ?

S: Region C.

T: Why ?

S: Because all positive numbers (positive divisors of 27 or positive prime numbers) are larger than -10 and, hence, inside the red string.

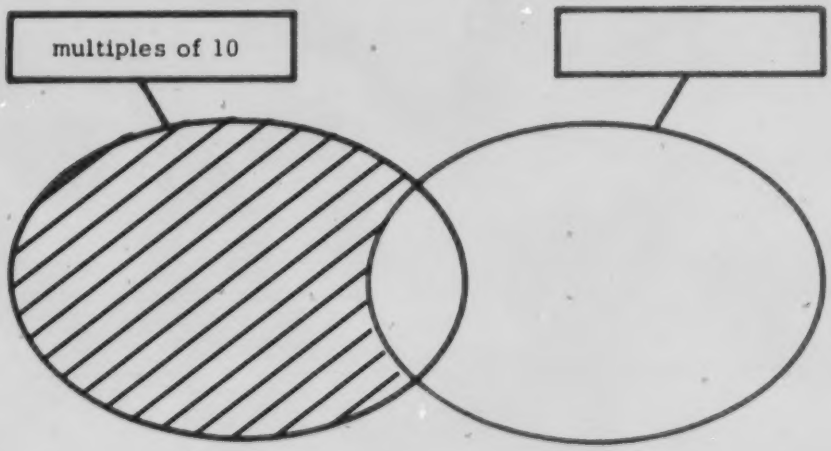
Hatch region C and check that the picture is correct for both possibilities for the blue string.

Erase the board before going on to Exercise 2.

5L

Exercise 2

Distribute individual String Game Analysis Pad sheets. Draw this string picture on the board.



T: Imagine that we are playing the String Game and we have determined that the red string is for the multiples of 10. Suppose we also have the clue that this region (point to the hatched region) is empty. What could the blue string be for ? Look at your list of possible string labels.

Let the students make suggestions and for each one have the class check whether or not it is correct. For example,

S: The blue string could be for the multiples of 2.

S: Yes. Every multiple of 10 is also a multiple of 2, so all the numbers in the red string would be in the middle region.

For incorrect suggestions, the class should find a number which would be in the hatched region. For example,

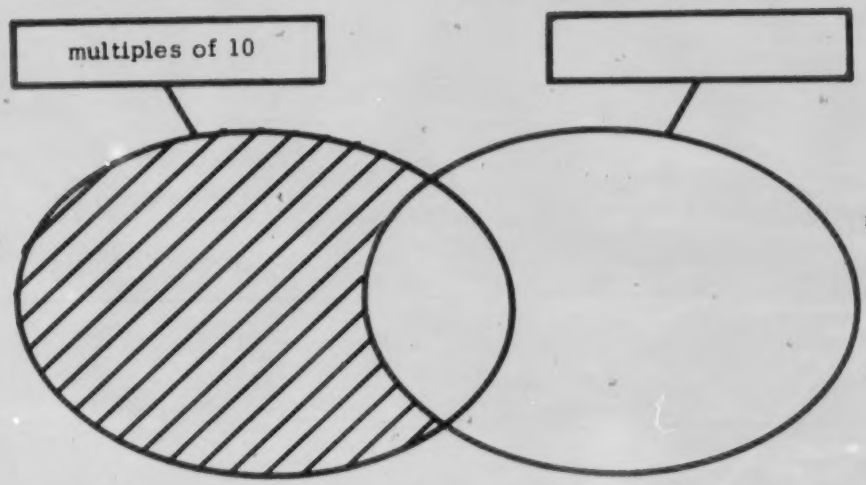
S: The blue string could be for the numbers larger than $^{-}10$.

S: No. ⁻10 (or ⁻80) is not larger than ⁻10 and is a multiple of 10, so it would be in the hatched region.

S: The blue string could be for the multiples of 4.

S: No. 10 (or 50) is not a multiple of 4 and is a multiple of 10, so it would be in the hatched region.

As the class finds the three possibilities* for the blue string, list them near the blue string.



* In order to find the third possibility, the class must understand that it is possible for both strings to have the same label. If your class has never discussed this situation in the past or played a game where the strings did have the same labels (see the Appendix: The String Game, example 5 of Version D, Variation #1, page 131), you will need to discuss the possibility at this time.

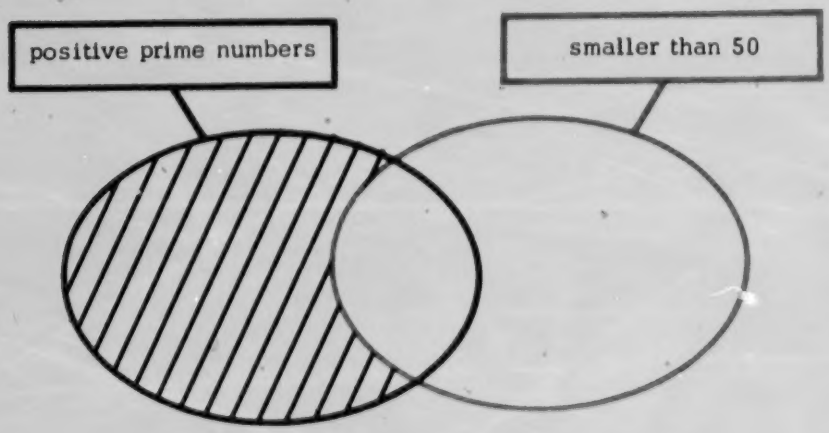
5L

Repeat this exercise with some of the following label changes. Start with the unboxed label. Correct responses are boxed.

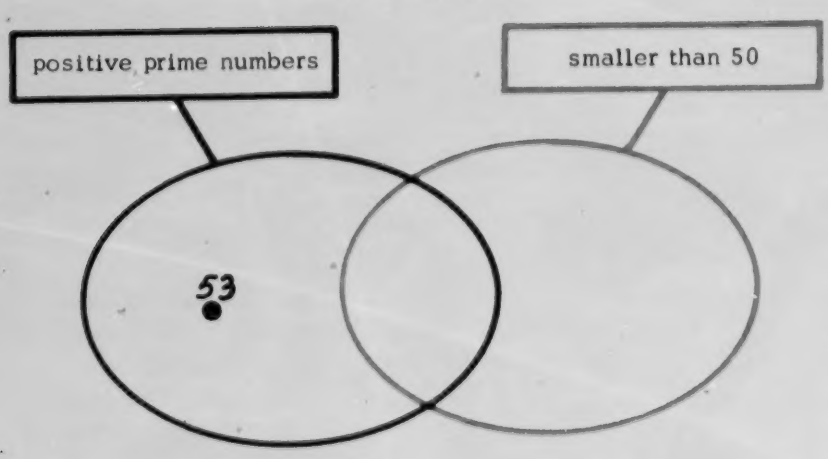
Red String	Blue String
positive divisors of 12	positive divisors of 12 or positive divisors of 24 or smaller than 50 or larger than $^{-}10$
smaller than $^{-}10$	smaller than $^{-}10$ or smaller than 50
larger than 50	larger than 50 or larger than $^{-}10$
multiples of 2 or multiples of 4 or multiples of 10	multiples of 2
larger than 50 or larger than $^{-}10$ or positive divisors of 12 or positive divisors of 18 or positive divisors of 20 or positive divisors of 24 or positive divisors of 27 or positive prime numbers	larger than $^{-}10$

NOTE: It is implicit in these exercises that we are only using numbers from the String Game. In some situations, regions which could be hatched because of this limited set of numbers would <u>not</u> be hatched if we could use any integer. We have avoided such situations in order not to suggest erroneous conclusions. An example of such a situation is illustrated below.

Using only String Game numbers



Using all integers



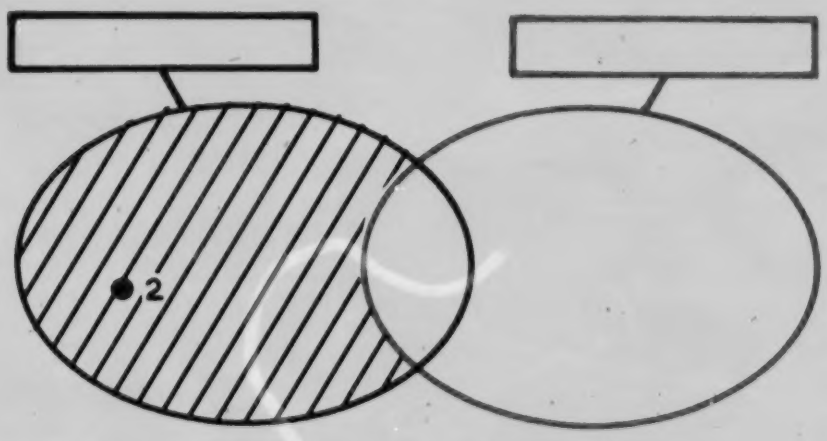
Erase the labels, but leave the string picture on the board, before going on to Exercise 3.

5 L

Exercise 3

T: Now, instead of this region (point to the hatched region) being empty, let's suppose there is exactly one number in that region; the number 2.

Put 2 in your picture.



T: What could the red and blue strings be for ?

Let students make suggestions and have the class check each suggestion. For example,

S: The red string could be for the multiples of 2 and the blue string for the multiples of 4.

S: No. 6 is a multiple of 2, but not a multiple of 4, so 6 would be in the hatched region.

S: The red string could be for the positive divisors of 18 and the blue string for the positive divisors of 27.

S: No. 18 is a positive divisor of 18, but not a positive divisor of 27, so 18 would be in the hatched region.

After some trial and error, your students will probably discover that this new clue determines the strings.

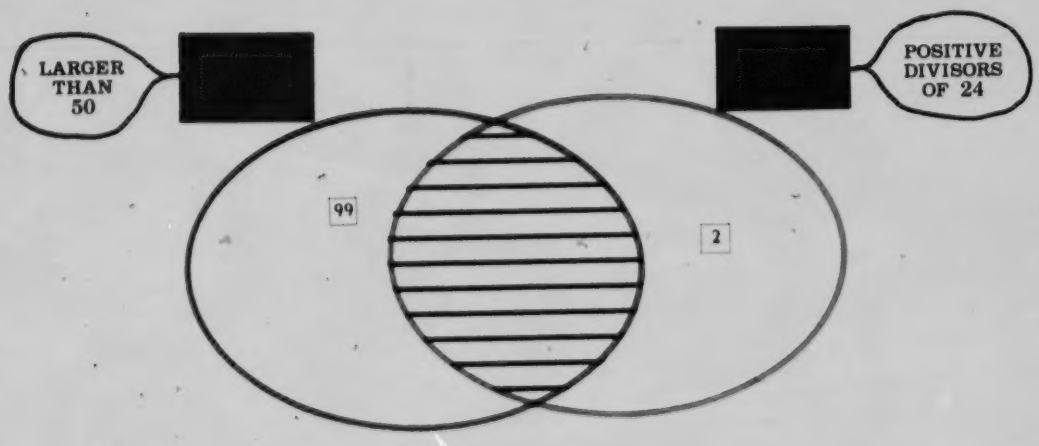
S: The red string is for the positive prime numbers and the blue string is for the odd numbers.

S: Yes. The only positive prime number which is not an odd number is 2.

If the analysis of this situation becomes too difficult and students appear frustrated, leave the exercise as a challenge puzzle.

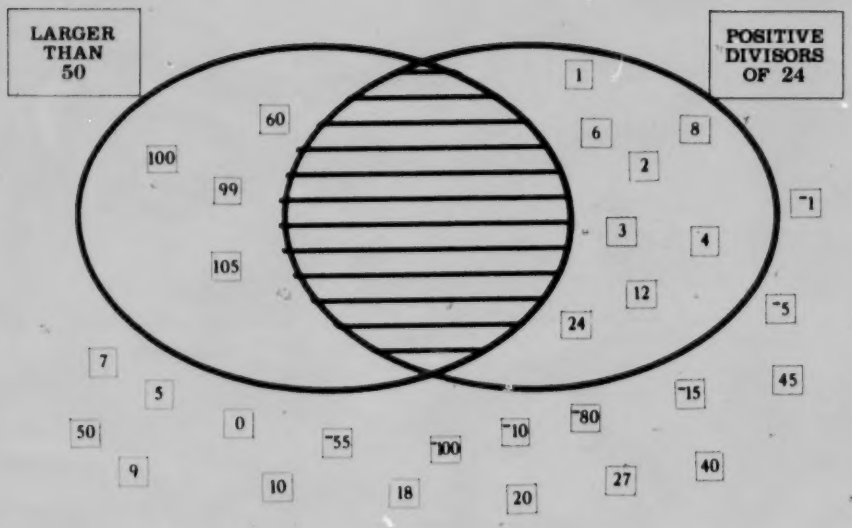
Play the String Game in any time that remains. An interesting starting situation you could use for the game is illustrated below. One clue is that the middle region is empty. Remind the class that the hatching should give some information about the strings.

Starting Situation



5L

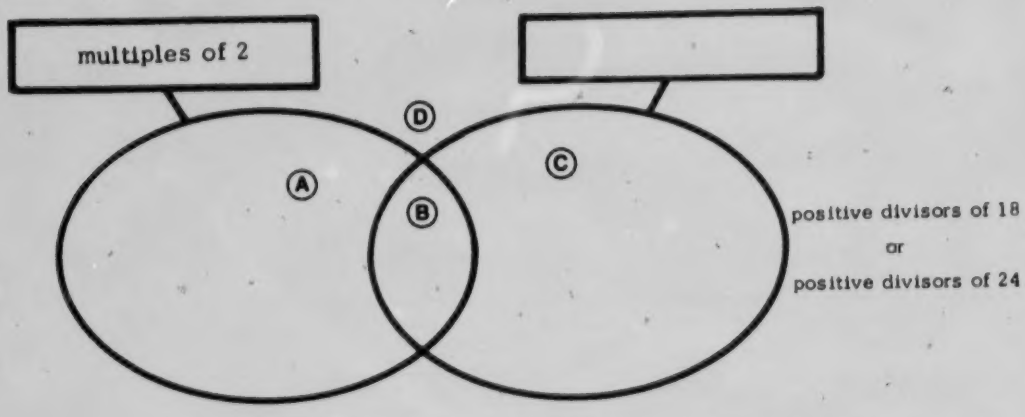
End of Game



2.3 String Game Analysis #3

Exercise 1

Draw this string picture on the board.



T: Imagine that we are playing the String Game and we have determined that the red string is for the multiples of 2 (point to the red string label). There are still two possibilities for the blue string. It could be for the positive divisors of 18 or for the positive divisors of 24.

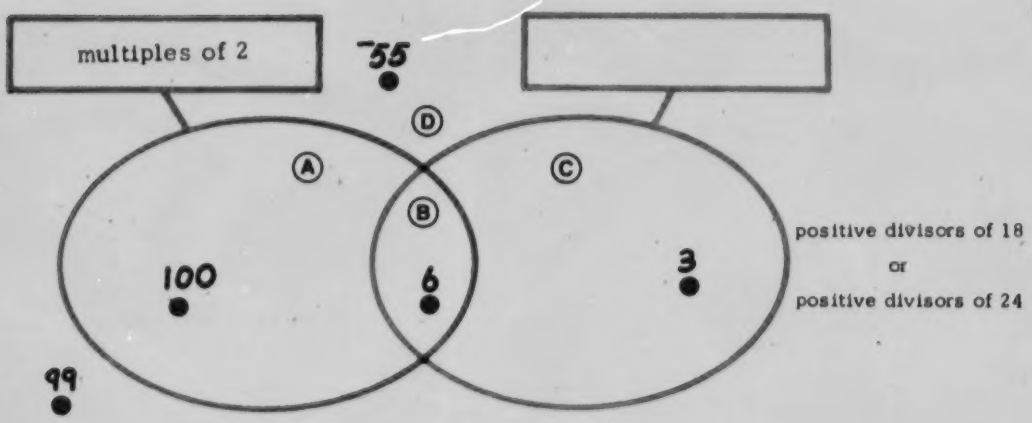
Write these labels on the board near the blue string, but not in the blue box. Aside from the picture, write this list of numbers on the board.

6, 4, 3, 9, 100, 18, -55, 99

T: Some of these numbers you can be sure where to put in the string picture even though the blue string is not determined. Who can show us the correct place for one of these numbers?

Let students respond, in any order they wish, that a number is in a certain region or that we can not be sure which of two regions a number is in. Encourage the class to check each response before continuing. Put the numbers that can be placed for sure in the correct regions in your picture. Next to those numbers

which cannot be placed for sure, write the names of possible regions in which they could be. Correct responses are indicated in this picture.



A or B C or D A or B

~~2~~, 4, ~~3~~, 9, ~~10~~, 18, ~~-3~~, ~~9~~

T: Suppose it is your turn in the String Game and you want to play with [4]. Where would you play it?

S: In region A.

NOTE: Region B would also be a good answer. If a student says region C or D, point out that then they are sure to get a "no" answer.

T: Okay; then if I say "yes", what do you know about the blue string?

S: It must be for the positive divisors of 18.

T: Why?

S: Because 4 is not a positive divisor of 18 and it is outside of the blue string.

S: The blue string cannot be for the positive divisors of 24, because 4 is a positive divisor of 24 and is not in the blue string.

T: Okay; suppose instead I say "no". What would you know about the blue string ?

S: It is for the positive divisors of 24, because then 4 must be in the region B, not in region A.

S: 4 is not a positive divisor of 18, so the blue string cannot be for the positive divisors of 18.

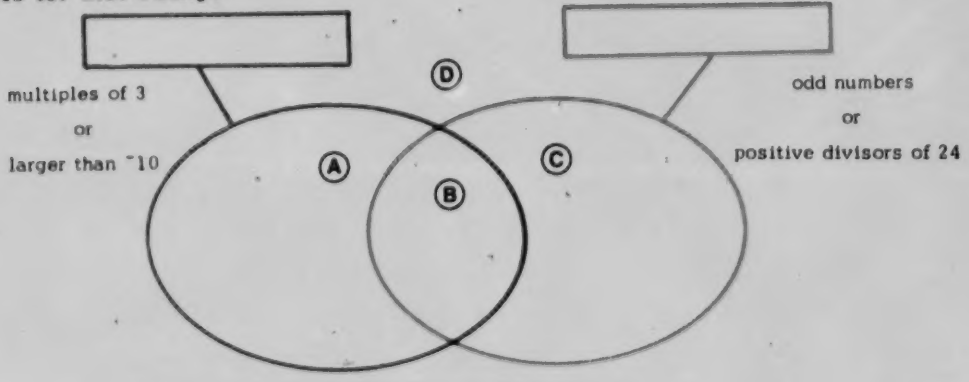
T: So, no matter how I answer, you will know what the blue string is for.

Repeat, asking students where they would play [9] and [18] .

Erase the board before going on to Exercise 2.

Exercise 2

Draw this string picture on the board. Near each string, write the two possible labels for that string.



8L

T: Imagine we are playing the String Game. Suppose there are two possibilities for the red string; it could be for the multiples of 3 or for the numbers larger than ⁻10. Also, there are two possibilities for the blue string; it could be for the odd numbers or for the positive divisors of 24.

Now, I will give you some statements. With this information, you must say whether the statement is true or false or you can't tell.

As you make each statement, put the number in the indicated region. If the statement is true, leave the number in the picture. If the statement is false or you can't tell, remove it from the picture. Encourage class discussion about each statement, and request an explanation about why the statement is true or false or you can't tell.

1. 3 is in region B.

(True. 3 is a multiple of 3 and larger than ⁻10; therefore, 3 is inside the red string. 3 is an odd number and a positive divisor of 24; therefore, 3 is inside the blue string. Being inside both strings, 3 must be in region B.)

2. ⁻80 is in region D.

(True. ⁻80 is not a multiple of 3, is not larger than ⁻10, is not odd, and is not a positive divisor of 24; therefore, ⁻80 is outside of both strings and, hence, in region D.)

3. 27 is in region C.

(False. 27 is a multiple of 3 and is larger than ⁻10; therefore, 27 is inside the red string and, hence, cannot be in region C.)

4. 40 is in region A.

(Can't tell. 40 is outside of the blue string, because it is not odd and not a positive divisor of 24. But 40 would be in region A if the red string were for the numbers larger than $^-10$, or it would be in region D if the red string were for the multiples of 3. So we can't tell for sure.)

5. 40 is in region D.

(Can't tell. See reasoning for #4 above.)

6. 12 is in region C.

(False. 12 must be inside the red string because it is a multiple of 3 and larger than $^-10$.)

7. 12 is in region B.

(Can't tell. 12 would be in region A if the blue string were for the odd numbers, or 12 would be in region B if the blue string were for the positive divisors of 24.)

8. $^-15$ is in region D.

(Can't tell. $^-15$ could be in any of the four regions as indicated by this table.

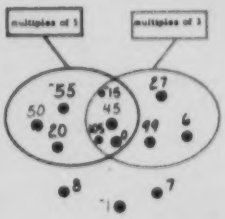
RED	BLUE	$^-15$ is in
multiples of 3	odd numbers	B
multiples of 3	positive divisors of 24	A
larger than $^-10$	odd numbers	C
larger than $^-10$	positive divisors of 24	D

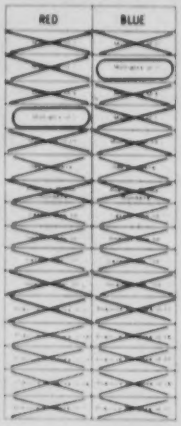
Worksheets L22, L23, L24, and L25 are available for the rest of the lesson. If any students have difficulty determining the strings on Worksheet L22, do the analysis collectively. After the strings have been determined, let students independently work on putting the numbers in the string picture.

Name _____ L 22 *

Cross out the labels that the strings cannot have. Some are done for you.

Complete the tags.



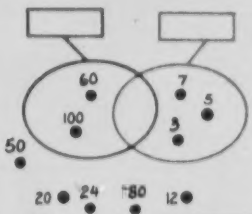


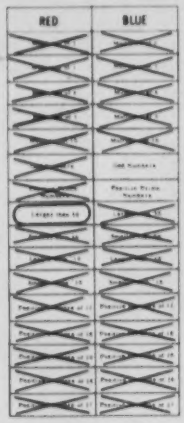
Put these numbers in the string picture.

-55; -15; 0; 6; 7; 8; 20; 27; 99; 105

Name _____ L 23 **

Cross out the labels that the strings cannot have. Some are done for you.





The label for the red string is [larger than 50]

The label for the blue string could be

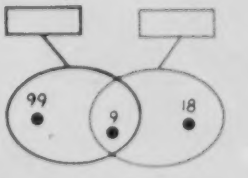
[positive prime numbers] or [odd numbers]

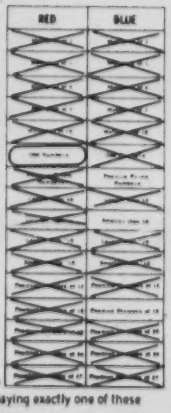
Exactly four of the numbers below cannot be put in the string picture because the blue string is not known. Circle these four numbers. Put all the other numbers in the string picture.

-80; (-15) (2) 3; 7; 24; 50; 60; (99) (105)

Name _____ L 24 ***

Cross out the labels that the strings cannot have.





The label for the red string is [odd numbers]

The label for the blue string could be

[smaller than 50] or [positive divisors of 18]

It is your turn in the String Game.
You want to find the blue string.

1) You can find the label for the blue string by playing exactly one of these numbers, even if you get a NO answer. Circle the number you should play.

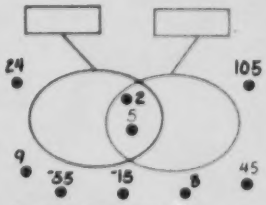
3; 105; 60; (7); 2;

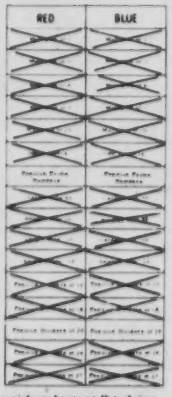
2) Repeat problem (1) but with these numbers.

(20) 100; 6; 55; -1;

Name _____ L 25 ****

Cross out the labels that the strings cannot have.





The label for the red string could be

[positive prime numbers] or [positive divisors of 20]

The label for the blue string could be

[positive prime numbers] or [positive divisors of 20]

Some of these numbers cannot be put in the string picture because the string labels are not known. Circle them. Put the others in the string picture.

-55; -15; 2; (3) (4) 8; 9; (10) 24; 105;

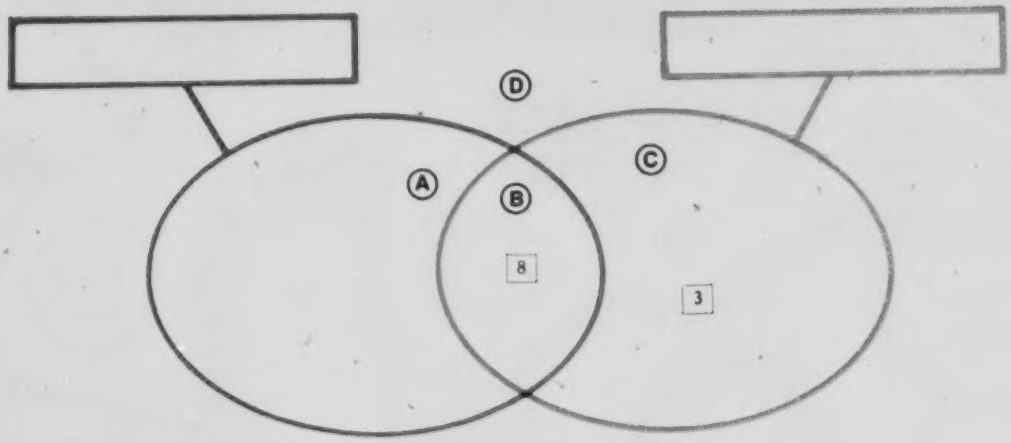
It is your turn in the String Game. By playing exactly one of these numbers you can find both of the string labels, even if you get a NO answer. Circle the number you should play.

(20) 100; 6; 55; -1;

2.4 String Game Analysis #4

Exercise 1

Before the lesson begins, set up your board for the String Game as illustrated below. Tape two demonstration analysis sheets (included in the Numerical String Game Kit) on the board; one to the left of the red string and one to the right of the blue string.



RED

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN 10
SMALLER THAN 50	SMALLER THAN 10	ODD NUMBERS	POSITIVE PRIME NUMBERS

BLUE

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN 10
SMALLER THAN 50	SMALLER THAN 10	ODD NUMBERS	POSITIVE PRIME NUMBERS

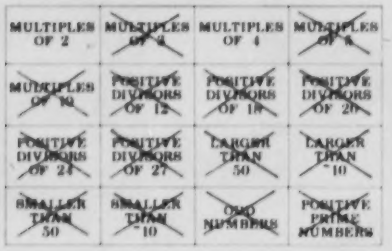
T: This is the starting situation for a String Game. What information do these two number clues give us about the red and blue strings ?

Let students suggest labels to cross off on the RED list and on the BLUE list. Before you cross off a particular label, encourage the student or someone else to give an explanation of why the string cannot have that label. The students

10 L

should eliminate all except two labels on the RED list and all except three labels on the BLUE list.

RED



BLUE

T: Now, I will give you some statements. With this information, you must say whether the statement is true or false or you can't tell. Be ready to explain your answer.

As you make each statement, put the number in the indicated region. If the statement is true, leave the number in the picture. If the statement is false or you can't tell, remove it from the picture. Encourage class discussion especially on statements about which there is some disagreement.

1. 12 is in region B. (True. 12 is a multiple of 2 and of 4; 12 is a positive divisor of 24 and is smaller than 50 and is larger than -10.)

2. 40 is in region C. (False. 40 is a multiple of 2 and of 4, so must be inside the red string.)

3. 40 is in region B. (Can't tell. 40 could be in region A or B. 40 is in region A if the blue string is for the positive divisors of 24; otherwise, 40 is in region B.)

4. -55 is in region D. (Can't tell. -55 could be in region C or D.)

137

5. 50 is in region D. (Can't tell. 50 could be in any one of the four regions as indicated in this table.

RED	BLUE	50 is in
multiples of 2	positive divisors of 24	A
multiples of 2	smaller than 50	A
multiples of 2	larger than -10	B
multiples of 4	positive divisors of 24	D
multiples of 4	smaller than 50	D
multiples of 4	larger than -10	C

6. 1 is in region C. (True. 1 is not a multiple of 2 or of 4; 1 is a positive divisor of 24 and is smaller than 50 and is larger than -10.)

7. 27 is in region A. (False. 27 is not a multiple of 2 or of 4.)

8. 27 is in region C. (Can't tell. 27 could be in region C or D.)

9. 24 is in region B. (True. See reasoning for #1 above.)

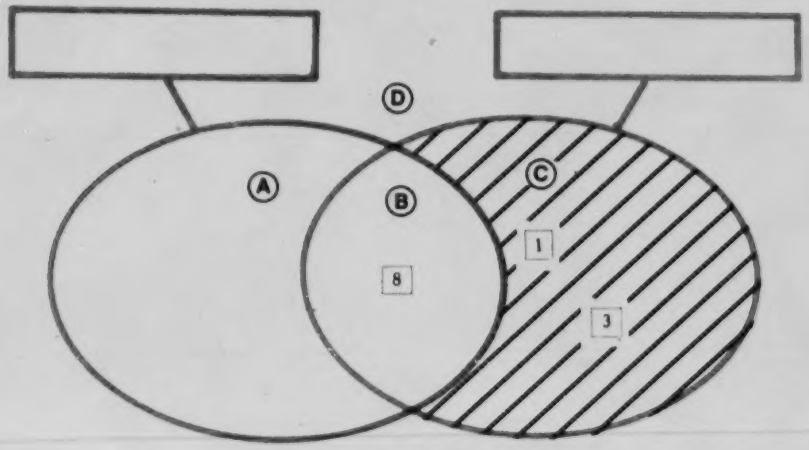
Leave your picture on the board for the next exercise. Take down all the number cards except [1], [3] and [8].

Exercise 2

T: We know that the red string could be for the multiples of 2 or for the multiples of 4. We know the blue string could be for the positive divisors of 24 or for the numbers smaller than 50 or for the numbers larger than -10. I am going to give you another clue: 1 and 3 are the only numbers in region C.

10 L

Indicate this clue on your picture by hatching region C.



T: With this new clue, you should be able to determine what the strings are for.

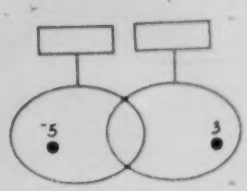
Give the class a couple minutes to think about this new clue. Then ask for some ideas about what the strings could be for. Check out each suggestion. After a while, the class should resolve that the red string is for the multiples of 2 and the blue string is for the positive divisors of 24. The table below shows why each of the other possibilities can be eliminated.

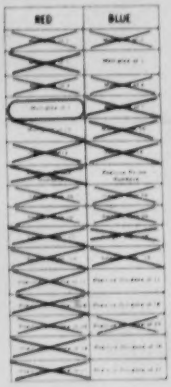
RED	BLUE	
multiples of 2	smaller than 50	27 (for example) would be in region C.
multiples of 2	larger than 10	27 (for example) would be in region C.
multiples of 4	positive divisors of 24	6 would be in region C.
multiples of 4	smaller than 50	27 (for example) would be in region C.
multiples of 4	larger than 10	27 (for example) would be in region C.

Distribute Worksheets L26 and L27. Let the students work independently for about 15 minutes. Then play the String Game for the rest of the lesson. Students who enjoy these String Game problems could work on Worksheets L28, L29, and L30 in their free time.

Name _____ L 26 *

Cross out the labels that the strings cannot have. Some are done for you.



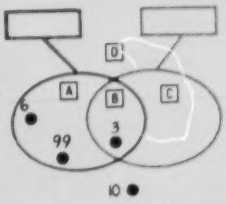


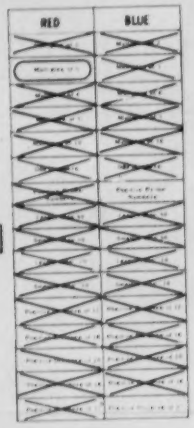
Circle one of the following: T (True)
F (False)
CT (Can't Tell)

1. The red string is for the numbers smaller than -10. T (F) CT
2. The blue string is for the positive prime numbers. T F (CT)
3. The red string is for the odd numbers. T (T) CT
4. The blue string is for the odd numbers. T (F) CT
5. The red string is for the multiples of 5. (T) F CT
6. The blue string is for the positive divisors of 27. T F (CT)

Name _____ L 27 **

Cross out the labels that the strings cannot have. Some are done for you.





The label for the red string is multiples of 3

The label for the blue string could be

positive prime numbers or positive divisors of 27

In the string picture, four regions are labelled
A, B, C, D.

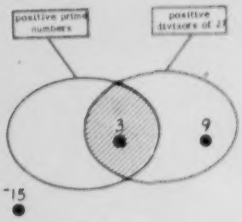
Circle one of the following: T (True)
F (False)
CT (Can't Tell)

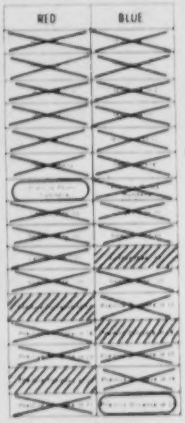
1) -15 is in region A. (T) F CT
2) 1 is in region C. T F (CT)
3) 0 is in region B. T (F) CT
4) 20 is in region A. T (F) CT
5) 20 is in region D. (T) F CT
6) 9 is in region B. T F (CT)
7) 105 is in region D. T (T) CT
8) 27 is in region A. T F (T)

Name _____ L 28 ***

Cross out the labels that the strings cannot have.
The hatching is also a clue.

Complete the tags.



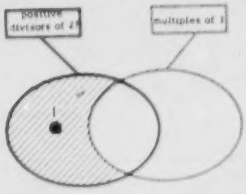


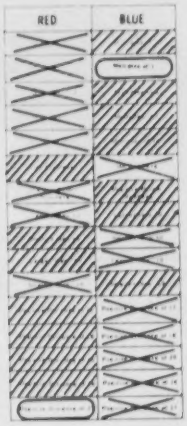
NOTE: The hatched labels in the lists were crossed out by using the hatching clue.

Name _____ L 29 ****

Cross out the labels that the strings cannot have.
The hatching is also a clue.

Complete the tags.

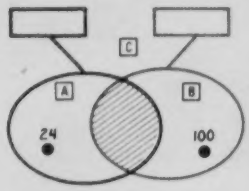


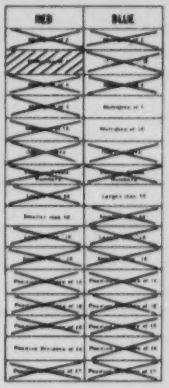


NOTE: The hatched labels in the lists were crossed out by using the hatching clue.

Name _____ L 30 *****

Cross out the labels that the strings cannot have.
The hatching is also a clue.





In this string picture, three regions are labeled:

A, B, C.

Circle one of the following: T (True)
F (False)
CT (Can't Tell)

1) 60 is in region A.	T	**(F)**	CT	6) 0 is in region C.	T	F	**(CT)**
2) 6 is in region A.	**(T)**	F	CT	7) 20 is in region A.	T	F	**(CT)**
3) 60 is in region B.	**(T)**	F	CT	8) 20 is in region B.	T	F	**(CT)**
4) 100 is in region B.	T	F	**(CT)**	9) 1 is in region A.	**(T)**	F	CT
5) 12 is in region C.	T	F	**(CT)**	10) 27 is in region B.	T	**(F)**	CT

NOTE: The hatched label in the RED list was crossed out by using the hatching clue.

APPENDIX: The String Game

The String Game is introduced in Part I (Grade Two) of the CSMP Upper Primary Grades Program and is used in many versions throughout the CSMP Elementary School Program. It gives students an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time it involves them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played using A-Blocks, but later on variations which make use of numbers make their appearance. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks or using numbers, and we begin the process of detailing the various possibilities open to you when you play the game. The different variations of the game described in this appendix are referred to in several of the lessons in this chapter. However, we suggest that you do not restrict your playing of the String Game just to the few lessons in this semester when it has been scheduled. It has been found that it is most beneficial (and enjoyable) for students if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

Equipment

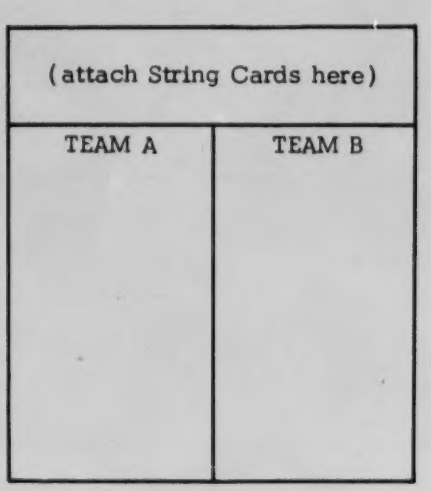
PLAYING BOARD

The most easily manageable equipment for this game depends upon your having a magnetic* chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards are not magnetic, try any portable chalkboard (grid board, etc.) that the school has available. If you do not have a magnetic chalkboard available, you can use your regular chalkboard.

* Strictly speaking the chalkboard itself is not magnetic, but rather magnet-sensitive; i.e., magnets adhere to it. However, for simplicity we call such boards "magnetic".

GAME BOARD

The Game Board is divided into three regions all illustrated below.



a) Magnetic: If you do locate a magnetic chalkboard, you will also need a magnetic or metallic Game Board. With a large magnetic chalkboard, say at least three chalkboard squares, the Game Board could be drawn on the board. However, if you have a relatively small (portable) magnetic chalkboard, then you will need to obtain a sheet of metal (minimum size: 60 cm by 80 cm) or locate a convenient metallic surface in the classroom such as the side of a file cabinet on which to put the Game Board. In this case draw the markings for the three regions of the Game Board on a large sheet of (easel pad) paper and tape this paper to your metallic surface.

b) Non-magnetic: If you do not have a magnetic chalkboard available for the playing board, then your Game Board could be a large piece of poster board (minimum size: 60 cm by 80 cm). Make the markings for the three regions of the Game Board with a heavy felt-tip pen.

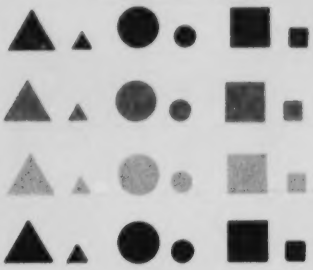
GAME PIECES and STRING CARDS

One set of game pieces and two sets of string cards are needed for each version of the game. One set of the string cards is the "face-up" set and is to be attached to the upper section of the Game Board - it is a constant reminder during the game of the possible labels for the strings. The other set is the "face-down" set.

Game Pieces

String Cards

String Game with A-Blocks



△	○	□	RED	BLUE	GREEN	YELLOW	BIG
NOT △	NOT ○	NOT □	NOT RED	NOT BLUE	NOT GREEN	NOT YELLOW	LITTLE

String Game with Numbers: Multiples and Order

-100	-80	-55	-20	-15	-12
-6	-5	-2	-1	0	1
7	10	15	24	27	30
40	45	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5	MULTIPLES OF 10
ODD NUMBERS	LARGER THAN 50	SMALLER THAN 50	LARGER THAN -10	SMALLER THAN -10

String Game with Numbers: Multiples, Divisors, and Order

-100	-80	-55	-15	-10	-5
-1	0	1	2	3	4
5	6	7	8	9	10
12	18	20	24	27	40
45	50	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN -10
SMALLER THAN 50	SMALLER THAN -10	ODD NUMBERS	POSITIVE PRIME NUMBERS

Game pieces and string cards for the String Game with Numbers, all versions, can be found in the Numerical String Game Kit. You will have to note which number cards and string cards to use with the particular version you are playing.

a) Magnetic: The game pieces (A-Blocks or number cards) can be magnetized by sticking to the back of each one a small piece of magnetic material. (Magnetic material is included in the Numerical String Game Kit, or it is available in many hobby and notions departments of stores.) To each string card in the "face-down" set, stick a small piece of magnetic material to the front of the card, taking care not to obscure what is written thereon.

b) Non-magnetic: Game pieces and string cards can be attached to the Game Board by means of loops of masking tape stuck to the backs. A "face-down" string card should have a loop of masking tape stuck to the front so that what is written on the card is not obscured. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if a loop of tape loses its stickiness, it can be replaced on the spot. An alternative is to use a small wad of a plastic caulking compound (Rope Caulk or Mortite, for example) in place of the loop of masking tape. This material will maintain its stickiness over a much longer period of time.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the playing board using two (or three) different colors. Next to each of these strings attach one string card from the face-down set. Stand the Game Board on the chalk tray, or have it near and convenient to the playing board. Share the game pieces between the two sections of the Game Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; rows 1, 2, 3 vs. rows 4, 5, 6; etc. Call one team "Team A" and the other "Team B". Before any student takes a turn, correctly place an equal number (at least one) of each team's game pieces on the string picture. This

removes from the first players the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by the number of pieces you place on the game board before the game begins.

Object of the Game

Each team tries to place all of its game pieces correctly (according to the face-down string cards) on the string picture. The winning team is the one which first places all of its game pieces correctly and identifies the face-down cards correctly, after playing according to the rules.

Rules of the Game

1) The game is played in silence. This is to avoid one student telling another where to play a certain game piece, or what the strings are for. Each student should have the opportunity to analyze the game alone. Infringement of this rule by anyone is penalized by the talker's team losing its next turn.

2) The teams alternate, and the members take turns within each team. A player comes to the board and selects a piece from his or her team's collection to place in one of the regions of the string picture.

3) You are the judge. If the piece is correctly placed, say "yes". The piece remains on the string picture and the player immediately has a second turn (no player may have more than two consecutive turns). If the piece is incorrectly placed, say "no". The player returns the piece to the team's unplayed collection and play passes to the other team.

   As an aid in judging, prepare a "crib-sheet" showing the correct position of each game piece or at least reminding you of what is on the face-down cards. If at any time you discover that you have made an error, say so immediately and rectify the mistake. Either move an incorrectly placed piece whose position you had approved to its correct region or replace on the string picture a correctly placed piece that had been rejected because you had disapproved its position.

4) When a team has correctly placed all of its pieces, the player who placed the last piece may thereupon attempt to identify each of the string cards.

If these are <u>all</u> correct, the team has won. If a mistake is made (even if it is only in the case of one of the string cards), simply indicate that the identification is incorrect and let the game continue.

5) If a team has exhausted its stock of game pieces, and the strings have not been identified, that team continues on its turn to attempt to identify the strings, while the other team works to place its game pieces.

<u>Version A: The String Game with A-Blocks</u>

This simplest version of the game uses 24 A-Blocks as game pieces and only nine string cards.

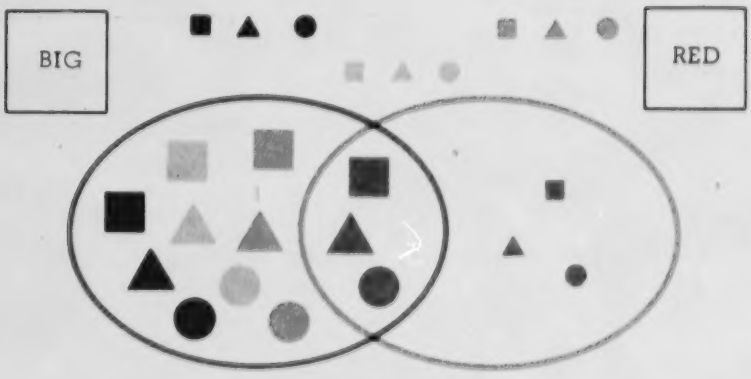
△	○	□	RED	BLUE	YELLOW	GREEN	BIG	LITTLE

(See the description under "Equipment" of game pieces and the <u>two</u> sets of string cards needed for the game.)

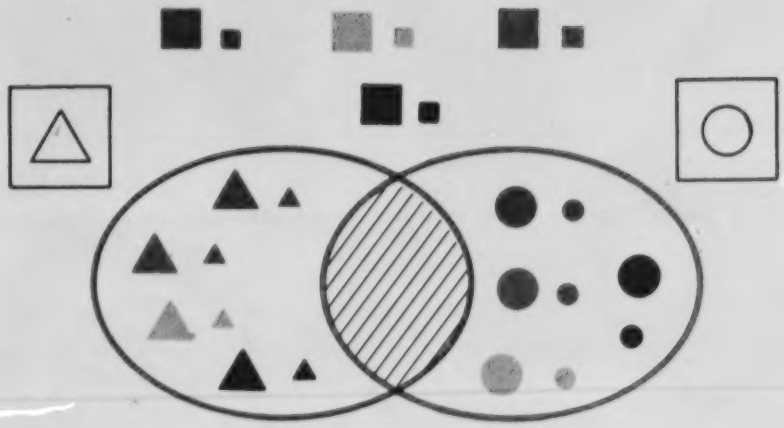
We provide here several example "crib-sheets" for variations of the game with two and three strings.

<u>Variation #1: Two Strings</u>

Example 1: No Empty Regions

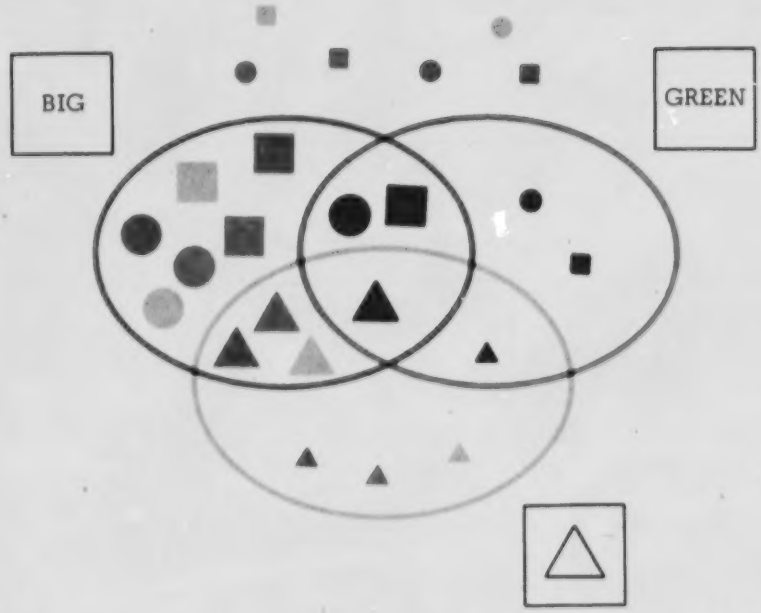


Example 2: One Empty Region

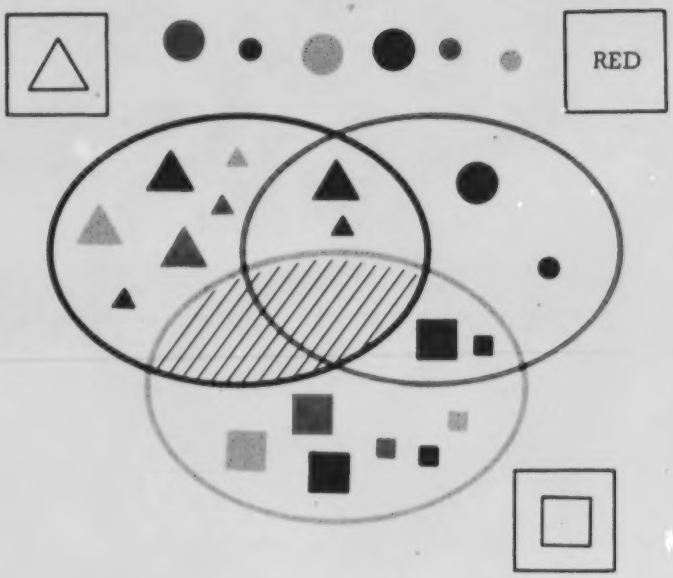


Variation #2: Three Strings

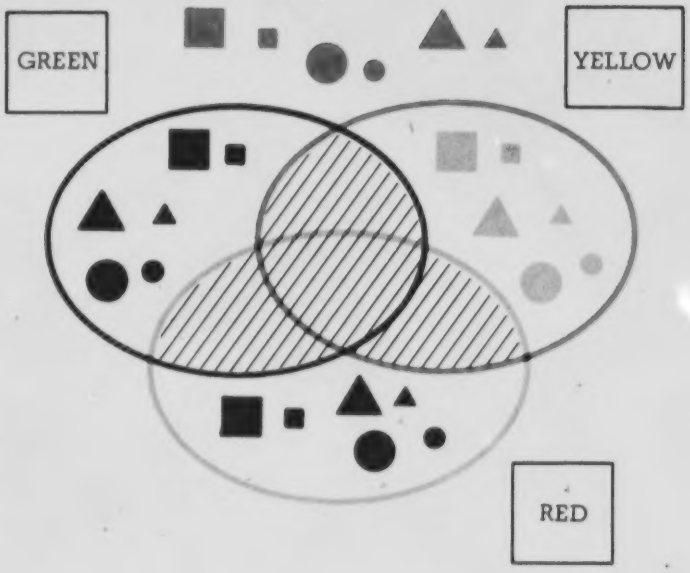
Example 1: No Empty Regions



Example 2: Two Empty Regions



Example 3: Four Empty Regions



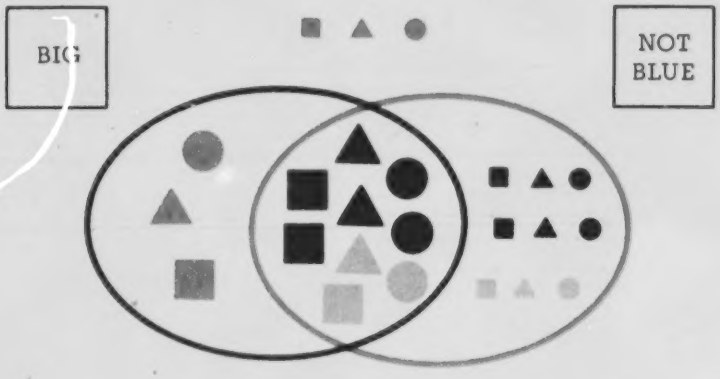
Version B: The String Game with A-Blocks, using "Not" Cards

A more complicated version of the game with A-Blocks uses all 16 of the string cards. (See the description under "Equipment" of game pieces and the two sets of string cards needed for the game with A-Blocks.)

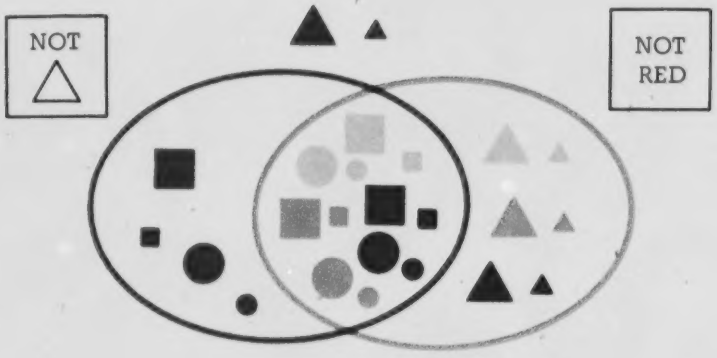
NOTE: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly for "NOT LITTLE" and "BIG"), such a "not" card is redundant. If a student questions why we do not include these possibilities, let the class discuss that it is unnecessary.

We provide here several example "crib-sheets" for this version of the String Game played with two strings.

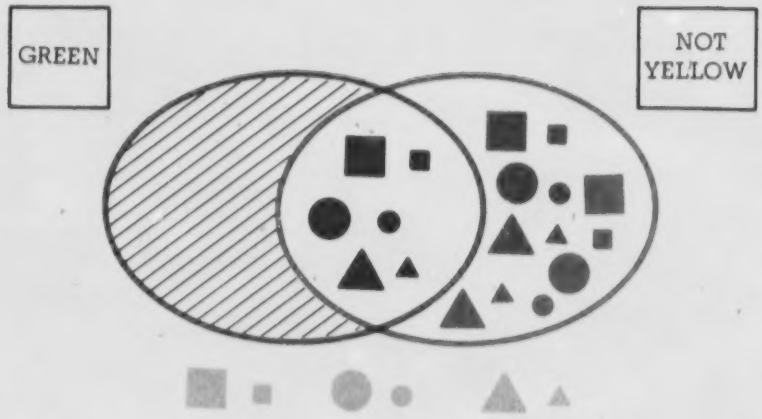
Example 1: No Empty Regions



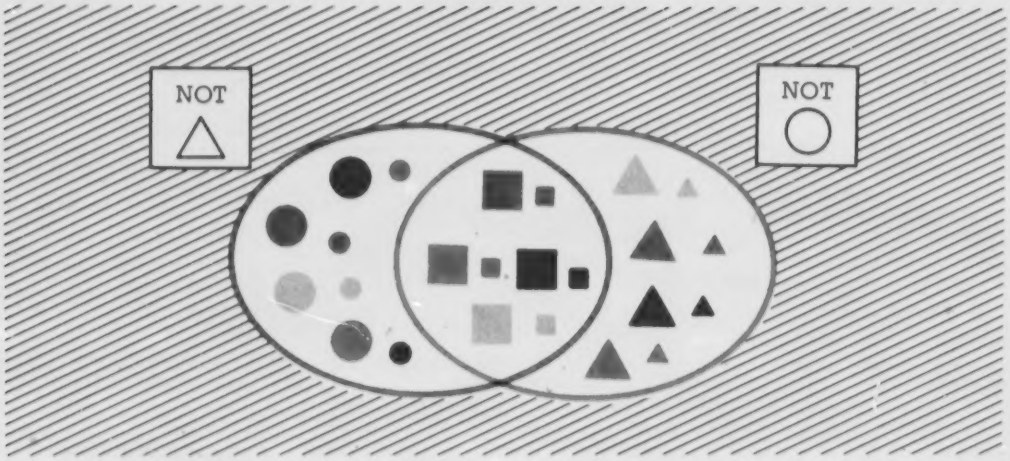
Example 2: No Empty Regions



Example 3: One Empty Region



Example 4: One Empty Region



Of course, by hatching the "outside" region of this diagram we mean to indicate that no game pieces can be placed there correctly. Strictly speaking, that region is not empty because, for example, the number 50 is in the outside region.

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your students.

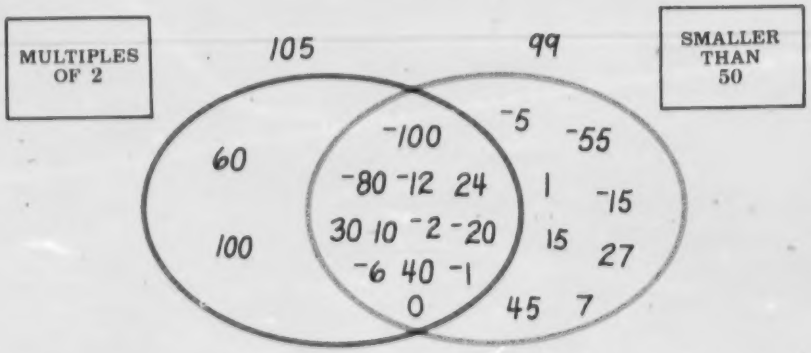
Version C: The String Game with Numbers; Multiples and Order

This version of the game uses 24 number cards as game pieces and two sets of ten string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string.

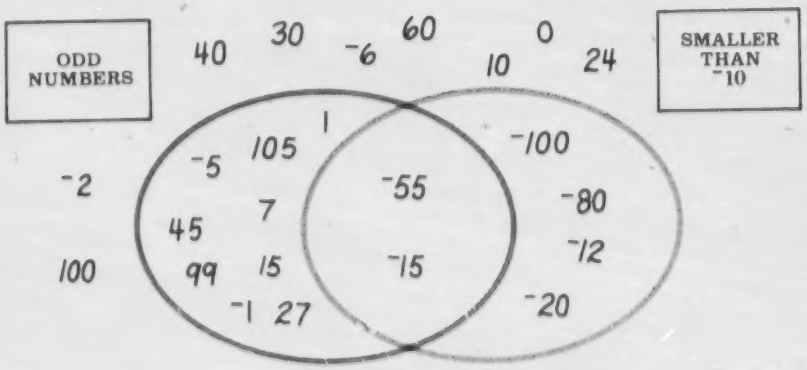
We provide here several example "crib-sheets" for variations of this game using two and three strings.

Variation #1: Two Strings

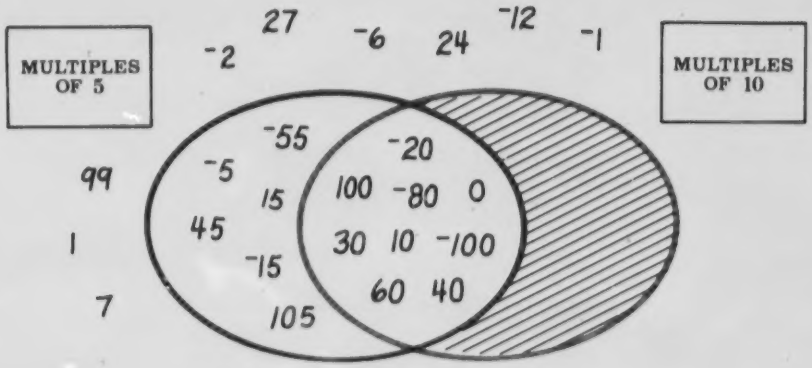
Example 1: No Empty Regions



Example 2: No Empty Regions



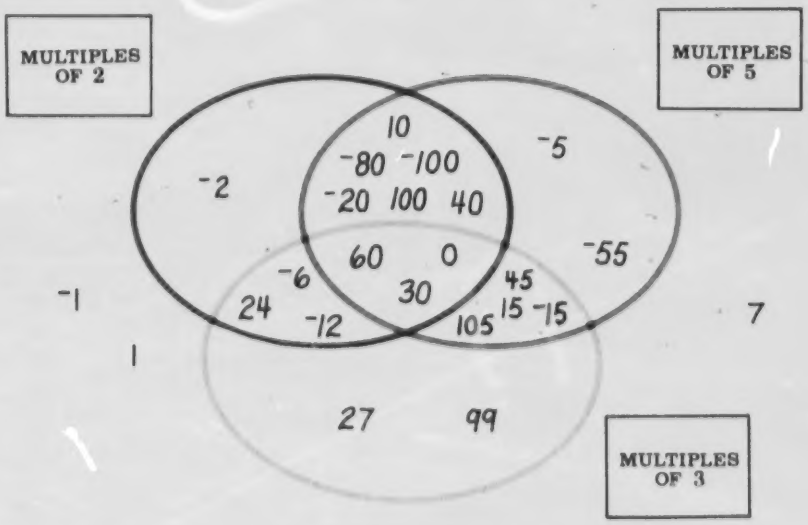
Example 3: One Empty Region



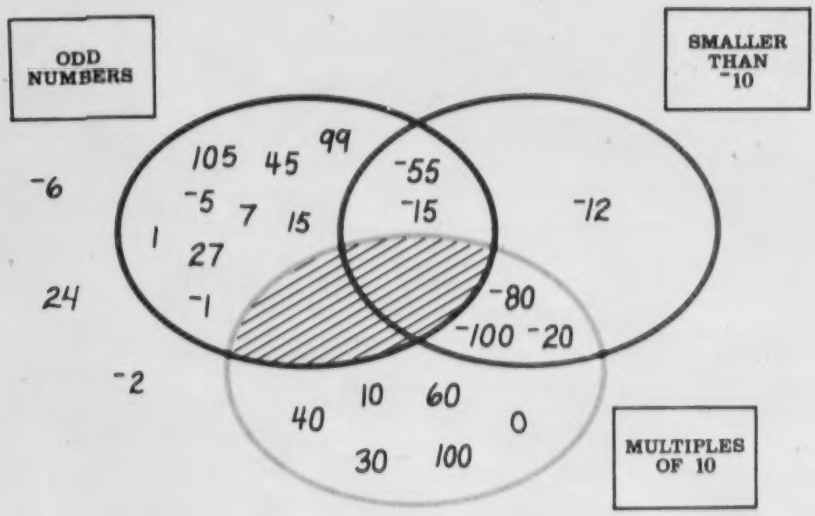
Variation #2: Three Strings

CAUTION: The jump in difficulty from the two-string variation to the three-string variation is much greater in the case of Version C of the String Game than it was in the case of Version A. Hence you should not expect to proceed to Variation #2 in this case as rapidly as you might have done with Version A.

Example 1: No Empty Regions



Example 2: Two Empty Regions

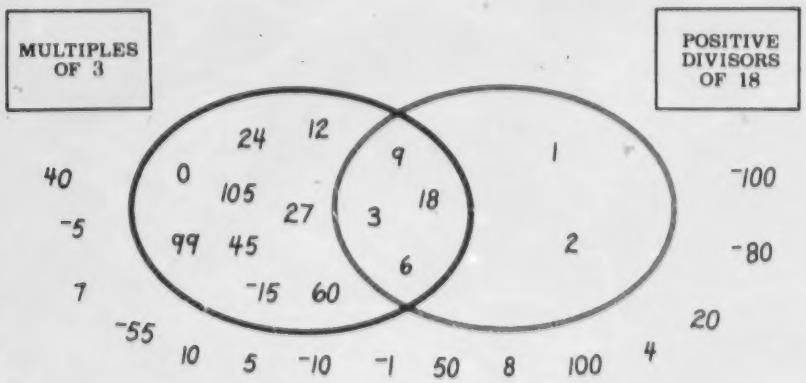


Version D: The String Game with Numbers: Multiples, Divisors, and Order

This version of the game uses 30 number cards as game pieces and two sets of 16 string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string. Two examples of this are provided in "crib-sheet" examples provided here; however, before using such a game the players should have previously understood that this was a possibility.

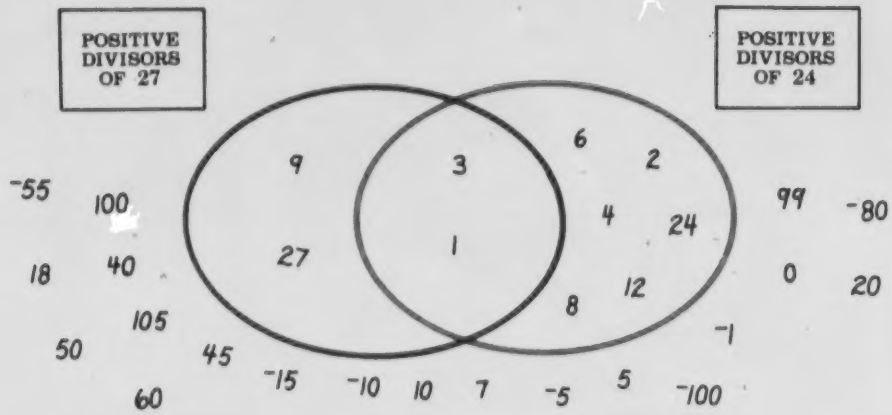
Variation #1: Two Strings

Example 1: No Empty Regions

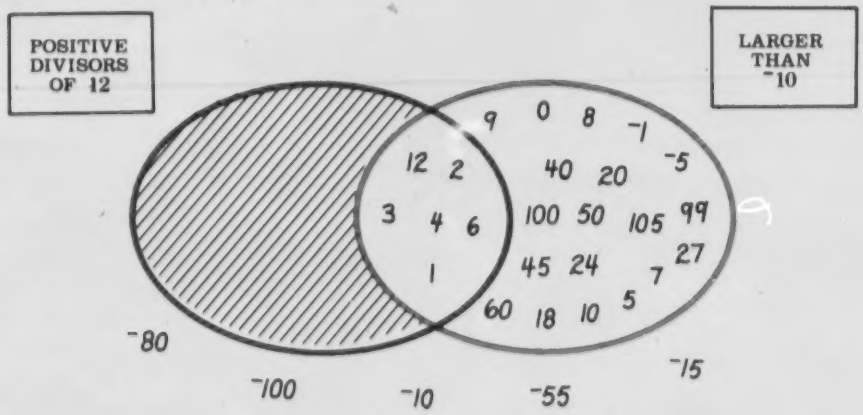


151

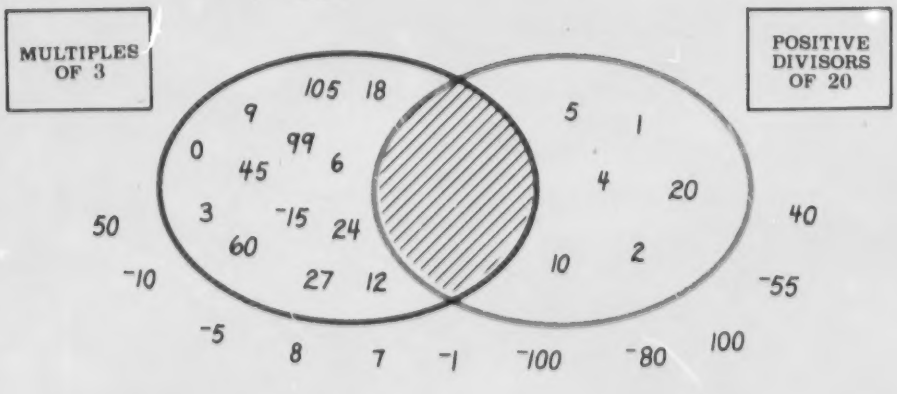
Example 2: No Empty Regions



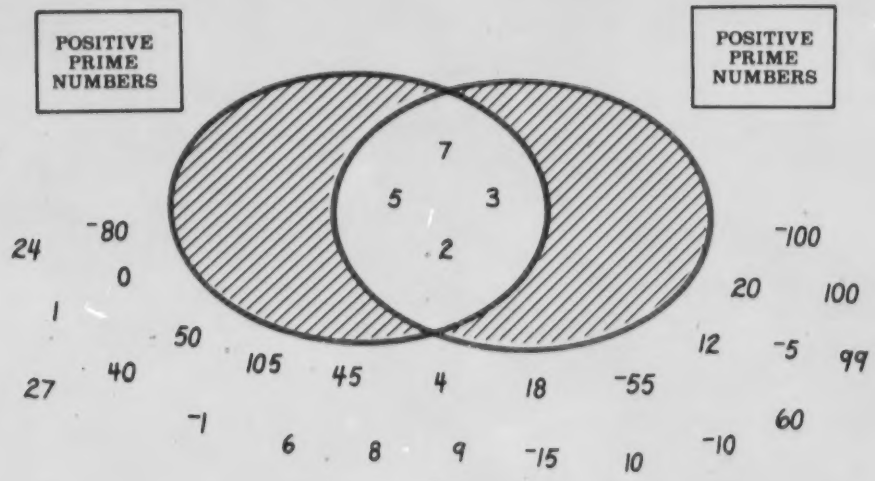
Example 3: One Empty Region



Example 4: One Empty Region



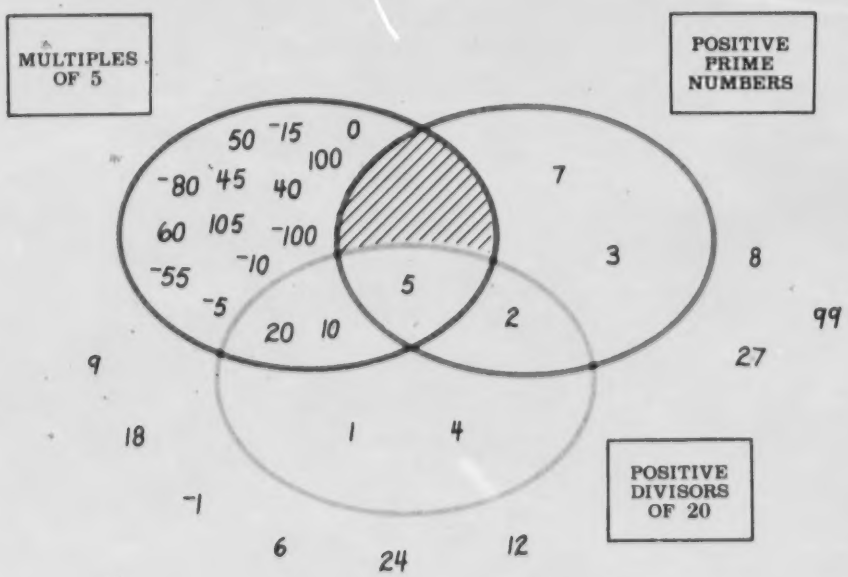
Example 5: Two Empty Regions



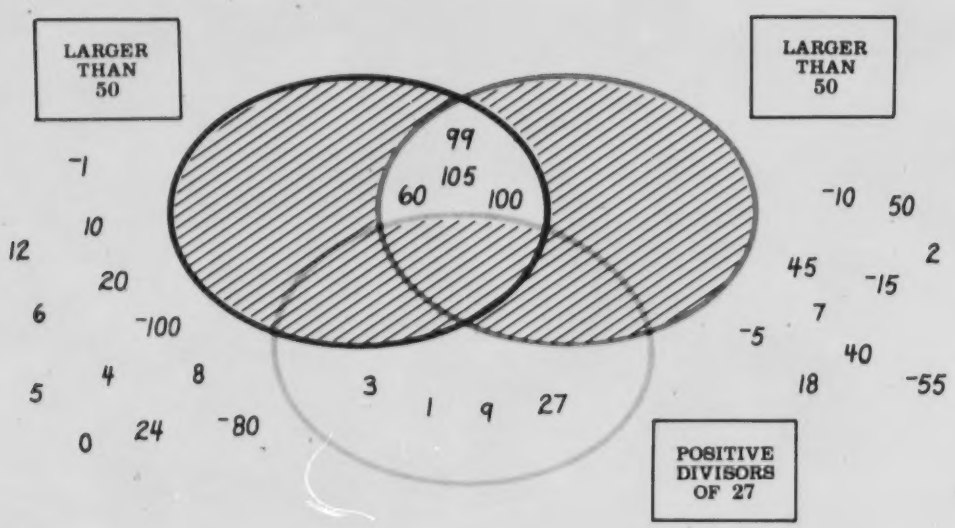
Variation #2: Three Strings

The warning concerning the jump from the two-string variation to the three-string variation on page 138 applies even more so to Version D.

Example 1: One Empty Region



Example 2: Five Empty Regions



Version E: The String Game with Special Scoring Rules

In this version of the game special scoring rules are introduced which determine the winning team. Both the String Game with numbers and with shapes can be played with these scoring rules, however, in the lessons of this book only the Numerical String Game is played this way.

Scoring

Each student is given a list of the possible string labels on a sheet from the individual String Game Analysis Pad. One round of the game is completed by having one player from each team take a turn. After each round the students have an opportunity to privately identify the face-down string cards. They do this by circling on their RED and BLUE lists what they believe is on the hidden tags and submitting it to the judge (teacher). However, each student has just one chance to make a private identification. If a player correctly identifies the string cards after a particular round, then points (according to the following schedule) are added to the player's team score.

Round 1 - 200 points	Round 3 - 100 points	Round 5 - 50 points
Round 2 - 150 points	Round 4 - 75 points	Round 6 - 25 points

After Round 6, play out the game without further private identifications. No more additional points can be scored by a team until the end of the game. The team that correctly identifies the string cards at the end of the game scores an additional 100 points plus 5 points for each unplayed game piece the opposing team has left on the Game Board. The winning team is the one with the greater total score.

Comments

1) Keep count of the rounds played on the board so that both you and the students know which round has just been completed.
2) After each round give the students time (perhaps 15-30 seconds) to study the picture, mark their lists, and if desired to submit them.
3) When the students submit their lists, write on each one the round after which it was turned in and sort them by teams. Do not return them to the students. You can total the team scores at the end of the game. Do not tell the students that their identifications are correct or incorrect, but encourage them to write their findings on a separate sheet of paper so as not to make mistakes when they play. They should continue to check their identifications during the remainder of the game.
4) It may be even more important to insist on silence during this kind of game, because an individual can have more influence on a team's winning or losing. We suggest you be strict about penalties for talking during the game.

Purpose

Scoring rules should not be imposed on the String Game until students have become very familiar and experienced in the play of the game. Many students will already be playing strategically, analyzing the clues given by correctly placed game pieces. Others may still be playing in a random way. The introduction of scoring rules will obviously favor the better students, and perhaps challenge them in a way the usual game is no longer doing. It provides

them the opportunity to more dramatically affect their team's chances of winning. For many students the scoring rules may provide incentive to begin analyzing the game and thinking about what information is given by a correctly played game piece.

We do not suggest that you play the String Game very often with the scoring rules. They do impose complication and give perhaps too much advantage to the better players. After playing the game with scoring rules a couple times, you should find that many students have become significantly better players. Returning to the usual way to playing the game, most students will continue to to be better players.

One advantage of using scoring rules when you play the String Game and have only a small amount of time is that you can stop the game before the end and have a reasonable way to determine the winning team. The winning team would be the one with the larger total number of points from identification of string cards.

GEOMETRY AND MEASUREMENT

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part III

Geometry and Measurement

CEMREL, Inc.

TABLE OF CONTENTS

SUGGESTED SCHEDULE FOR G - LESSONS iv

SUGGESTED CHRONOLOGICAL ORDER OF G - LESSONS vii

INTRODUCTION ix

CHAPTER 1: MEASUREMENT 1

1.1 Length #1 5

Worksheets G01, G02 11

1.2 Area #1 12

Worksheets G03, G04, G05, G06 15

1.3 Length #2 16

Worksheets G07, G08, G09 18

1.4 Area #2 19

Worksheet G10 31

1.5 Area #3 32

Worksheet G11 39

1.6 Area #4 40

Worksheet G12 43

CHAPTER 2: PARALLELISM 45

2.1 Projection #1 50

Worksheets G13, G14 59

2.2 Projection #2 60

Worksheets G15, G16, G17, G18 67

2.3 Couples and Equipollence 68

2.4 Translation #1 72

Worksheets G19, G20, G21, G22, G23 77

2.5 Translation #2 79

2.6 Translation #3 82

Worksheets G24, G25, G26 84

2.7 Lattice Construction 85

SUGGESTED SCHEDULE FOR G - LESSONS

WEEK	Geometry and Measurement	Materials
1	ADJUSTMENT DAY	
2	1G G1.1 Length #1 p. 5	Teacher: 30 cm ruler Student: 30 cm ruler; pencil; paper; Worksheets G01, G02
3	2G G2.1 Projection #1 p. 50	T: Small rubber ball; Demonstration translator; colored chalk S: Translator; colored pencils; Worksheets G13, G14
4	3G G1.2 Area #1 p. 12	T: Grid board; colored chalk S: Colored pencils; Worksheets G03, G04, G05, G06
5	ADJUSTMENT DAY	
6	4G G2.2 Projection #2 p. 60	T: Small rubber ball; Demonstration translator; colored chalk S: Translator; colored pencils; Worksheets G15, G16, G17 G18
7	5G G2.3 Couples and Equipollence p. 68	T: Demonstration translator; colored chalk S: Colored pencils; large sheets of unlined paper; translator
8	6G G1.3 Length #2 p. 16	T: 30 cm ruler S: 30 cm ruler; pencil; Worksheets G07, G08, G09
9	ADJUSTMENT DAY	

SUGGESTED SCHEDULE FOR G - LESSONS

WEEK	Geometry and Measurement	Materials
10	7G G2.4 Translation #1 p. 72	T: Demonstration translator; colored chalk S: Translator, colored pencils; Worksheets G19, G20, G21, G22, G23
11	8G G2.5 Translation #2 p. 79	T: Demonstration translator; colored chalk S: Translator; colored pencils; paper
12	9G G2.6 Translation #3 p. 82	T: Demonstration translator; colored chalk S: Translator; colored pencils; Worksheets G24, G25, G26
13	ADJUSTMENT DAY	
14	10G G1.4 Area #2 p. 19	T: Grid board; colored chalk S: Colored pencils; pencil; paper; hand-calculator (optional); Worksheet G10
15	11G G1.5 Area #3 p. 32	T: Grid board; colored chalk S: Colored pencils; pencil; paper; hand-calculator (optional); Worksheet G11
16	12G G1.6 Area #4 p. 40	T: Colored chalk S: Colored pencils; pencil; paper; hand-calculator (optional); Worksheet G12
17	ADJUSTMENT DAY	
18	13G G2.7 Lattice Construction p. 85	T: Demonstration translator; colored chalk S: Translator; large sheets of unlined paper; colored pencils

SUGGESTED CHRONOLOGICAL ORDER OF G - LESSONS

Lesson One	G1.1 - Length #1	5
Lesson Two	G2.1 - Projection #1	50
Lesson Three	G1.2 - Area #1	12
Lesson Four	G2.2 - Projection #2	60
Lesson Five	G2.3 - Couples and Equipollence . . .	68
Lesson Six	G1.3 - Length #2	16
Lesson Seven	G2.4 - Translation #1	72
Lesson Eight	G2.5 - Translation #2	79
Lesson Nine	G2.6 - Translation #3	82
Lesson Ten	G1.4 - Area #2	19
Lesson Eleven	G1.5 - Area #3	32
Lesson Twelve	G1.6 - Area #4	40
Lesson Thirteen	G2.7 - Lattice Construction	85

GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as number and calculations) and physical concepts (such as length, area, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense.

A benefit of early development of geometric thinking is that it provides alternative opportunities for "slow learners". Children develop number concepts at individual rates. Relative lateness of a child's numerical development often signifies a difficulty in remembering number facts. A common response to this problem is assignment of arithmetic drill, which often serves to confirm the diagnosis of slowness. However, numerical development may not be an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

The lessons of this strand provide an opportunity to explore geometric notions informally. The activities are varied to meet other areas of the program and are diffused over a long period of time. As a natural consequence of their involvement in the activities, the students will develop knowledge and skills.

But the effects of such an informal program must be measured in their appearance in subsequent contexts. No goals for mastery can be posted here; none could be certified. The focus of this strand is experience.

x

1

MEASUREMENT

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area and perimeter or for comparison of standard units, the lessons of this chapter provide rich, open-ended experiences which relate measurement to other areas of the mathematics curriculum. Direct experience of the tools and concepts is central. Skills and mastery of measurement tasks should develop naturally.

This chapter provides more practice in using the ruler to measure the length of line segments and to verify estimates of lengths. The problems require the student to construct zigzags under several types of restrictions and to measure perimeters of shapes to discover a message. Dissection and embedding methods for computing the areas of complex regions on a grid reinforce the student's sense of spatial relations between parts and whole. The worksheets offer mosaics in which to discover a design by coloring the pieces according to area-oriented instructions. Three lessons provide experience in calculating the area of a rectangle by multiplying the lengths of adjacent sides. In these lessons, the problems are posed as that of several farmers who will construct rectangular pens under differing constraints and who each wants a solution of largest possible area. Each of the problems has a best solution, in the search for which a student will incidentally find the areas of many rectangles.

The lessons in this chapter are supplemented regularly with pages in the workbook series, A Variety of Problems 1, 2, 3, 4, 5, and 6.

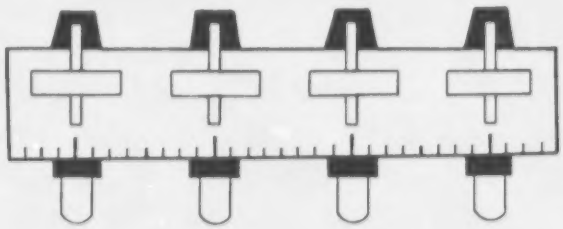
NOTE ON GRIDS

In three lessons in this chapter, as well as many lessons in the other parts of the CSMP Intermediate Grades Program, you will need a large grid board for demonstrations. A portable chalkboard with a permanent grid is convenient and efficient. Squares with eight centimeter sides are ideal. However, if

such a grid board is not available to you, there are several alternatives.

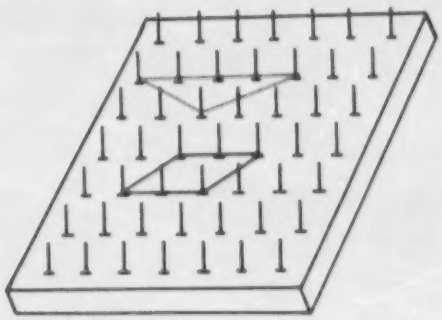
Printed grids on large sheets of paper* are provided with full classroom sets of materials for CSMP Mathematics for the Intermediate Grades, Part III. They can be taped to the board and discarded after use. Or one could be mounted on a board and covered with clear contact paper to provide a protective surface. Erasable wax crayon can then be used for drawings.

Another possibility is to carefully draw a grid on the chalkboard each time. The disadvantage of this technique is that inevitably the grid lines are partially erased each time a drawing is altered. However, one can make a tool for drawing grids. Use a music staff drawer with chalk in the first, third, and fifth positions. Or proceed as follows: clip four pen caps to a ruler at eight centimeter intervals and secure them with tape. Pen caps are usually of a diameter to hold a piece of chalk securely.



* These printed grids are packaged under the title "Robot Walk Grid". The title is from lessons in the CSMP Kindergarten Program and has nothing to do with the lessons in this chapter.

For lessons in which shapes are to be constructed for area or perimeter calculations, a geoboard often surpasses a grid board in usefulness. A geoboard consists of a grid board with nails or pegs protruding at the corners of the grid.



Instead of drawing lines, one uses rubber bands to construct shapes. Rubber bands are quickly and easily manuevered to facilitate the development of a lesson. In constructing a geoboard, use escutcheon or finishing nails. Let them protrude about two centimeters above the surface of the board. It is not necessary to draw a grid on the geoboard, but sufficient to mark each corner with a spot about one centimeter in diameter. The spots must be visible throughout the room. Omitting the lines of the grid emphasizes the shapes by focusing attention on the rubber bands.

Individual geoboards are compelling tools for the students and provocative in the readiness with which problems can be posed and attacked. Several models are available commercially. They can also be constructed easily. Make a master of a grid for duplication. Squares 3 cm by 3 cm are a good size for individual geoboards of six squares by six squares. Tape a copy of the grid to a piece of one inch plank. Hammer nails at the corners of the grid. Use a pair of pliers as a spacer to ensure uniformity of the nails' heights. Tear off the grid paper. There is no need to paint a grid or spots at the nails as on the demonstration geoboard, since the user is close enough to the geoboard for the nails themselves to define the grid. Use rubber bands of several colors; and for a demonstration geoboard, be sure to have some very large rubber bands.

CAPSULE LESSON SUMMARIES

Section 1.1: Introduce a new metric unit of length, the decimeter. Practice using the ruler to measure the length of line segments in centimeters and millimeters. Find the perimeters of shapes to the nearest centimeter.
(Worksheets G01, G02)

Section 1.2: Find the areas of shapes on a grid by decomposing them into triangles and rectangles, and by enclosing them in rectangles.
(Worksheets G03, G04, G05, G06)

Section 1.3: Practice using the ruler by constructing zigzags according to prescriptions.
(Worksheets G07, G08, G09)

Section 1.4: Find areas of rectangles by multiplication. Investigate the problem of finding the rectangle of largest area among those of perimeter 100 m.
(Worksheet G10)

Section 1.5: Find the largest possible area for a rectangular pen which a farmer can construct using a stream for one side and 100 m of fencing for the other three sides.
(Worksheet G11)

Section 1.6: Find the largest possible total area for two adjacent rectangular pens which another farmer can construct using a stream for one side of both pens and 100 m of fencing for the other sides.
(Worksheet G12)

1.1 Length #1

This lesson provides practice with the ruler in measuring the lengths of line segments. Metric units are used for all measurements and this lesson introduces a new metric unit of length; the decimeter. Before the lesson begins, be sure each student has a 30 cm ruler graduated in centimeters and millimeters.

Begin with a quick review of the use of the ruler, its markings, and the terms "centimeter" and "millimeter". Emphasize the relation "10 mm = 1 cm". Write all of this information on the board and ask the students to read it.

millimeter

centimeter

10 mm = 1 cm

Review the use of the decimal point in writing lengths as follows:

T: Suppose that you measure a line segment and find that its length is 12 centimeters and 7 millimeters. You might write "12 cm and 7 mm" for the length, but we usually write lengths in a shorter form. We write "12.7 cm" instead. This is an idea that comes from the way we write amounts of money. Ten dimes equal one dollar and ten millimeters equal one centimeter. 12 dollars and 7 dimes is written "$12.70" or "12.7 dollars", and 12 cm and 7 mm is written "12.7 cm".

Draw the following enlargement on the board.

1G

Point in turn to the marks representing 9.4 cm, 9.7 cm, 9.2 cm, and 9.5 cm. Each time ask a student to read the length represented and write that length on the board. Continue until the students have recalled the procedure of interpreting the marks.

T: There is another name for ten centimeters. Ten centimeters is the same as one decimeter. The symbol for decimeter is "dm".

Add this information to what is already on the board. Ask the students to read the word "decimeter" and the statement "10 cm = 1 dm".

millimeter
centimeter
decimeter

10 mm = 1 cm
10 cm = 1 dm

T: Everyone hold up two fingers about one centimeter apart.

Look around the room to verify and commend those who are approximately correct and to gently prod those who are far off.

NOTE: The following illustration shows an easy way to estimate 1 cm.

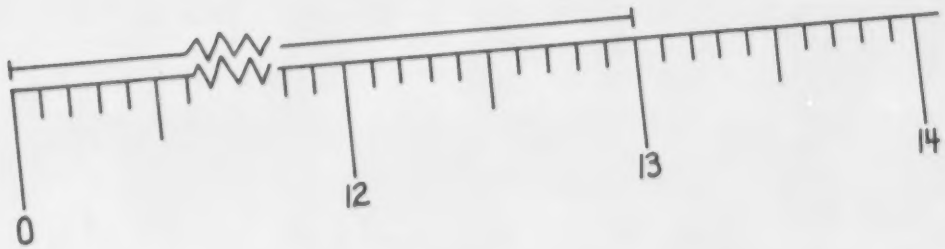
T: Now everyone hold two fingers about one decimeter apart.

Again, commend approximately correct displays and help to adjust incorrect displays. Remind the class that one decimeter is the same as ten centimeters.

T: There is an easy way to remember a decimeter.

Explain and demonstrate the technique of the following illustration.

Draw the following enlargement on the board.



T: Pretend that the longer marks on this ruler are one centimeter apart. Run your finger along the portion of the line segment above the ruler ending at the right endpoint.

T: One end of the ruler is exactly at one end of the line segment (point to the zero mark). How long is this line segment?

S: 13 cm.

T: How long in millimeters?

S: 130 mm.

1G

Write this statement on the board.

$$13.0 \text{ cm} = 130 \text{ mm}$$

T: How long in decimeters?

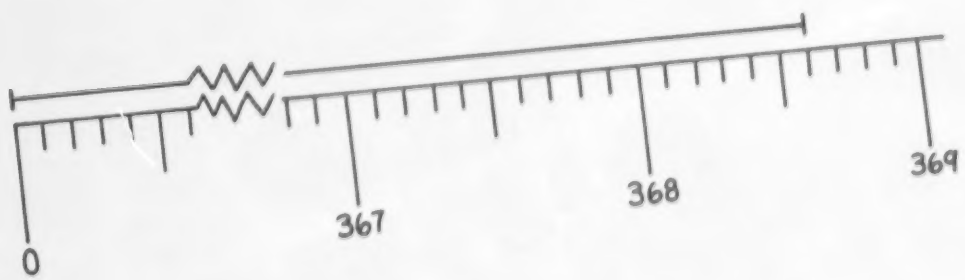
S: 1.3 dm.

Add this information to what is already on the board.

$$1.30 \text{ dm} = 13.0 \text{ cm} = 130 \text{ mm}$$

NOTE: Of course, 1.30 dm and 13.0 cm may more naturally be written "1.3 dm" and "13 cm", respectively. Use of the "0" in each of these serves only to emphasize the equivalences.

Change the notation on your drawing and the location of the right endpoint of the line segment.



175

Repeat the dialogue above with the altered drawing. Record the statements corresponding to this drawing below those of the first drawing.

1.30 dm = 13.0 cm = 130 mm

36.86 dm = 368.6 cm = 3,686 mm

There is a better way of writing large numbers of decimeters. We use a unit that you already know, the meter. Ten decimeters is the same as one meter.

Add this information to your lists of metric units and statements of relations among them.

millimeter
centimeter
decimeter
meter

10 mm = 1 cm
10 cm = 1 dm
10 dm = 1 m

T: How many centimeters in a meter?

S: 100. Ten centimeters in one decimeter, ten decimeters in one meter, and 10 × 10 = 100.

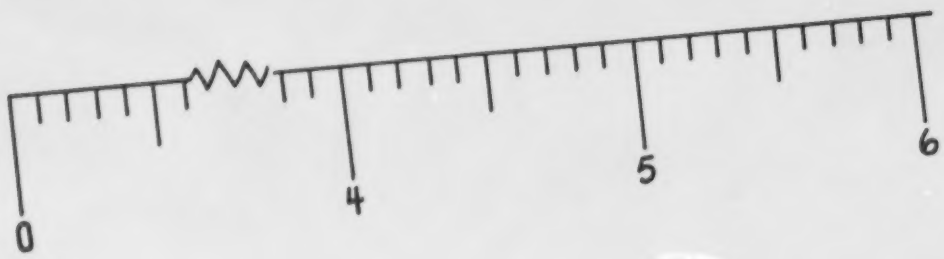
Go back to the record of the collective measuring of line segments and ask students to convert each measurement in turn to its equivalent in meters.

0.130 m = 1.30 dm = 13.0 cm = 130 mm

3.686 m = 36.86 dm = 368.6 cm = 3,686 mm

1G

Relabel your ruler drawing.



T: Now we will measure some line segments to the nearest centimeter. "The nearest centimeter" simply means that we find the nearest centimeter mark to the endpoint of the line segment when we measure its length.

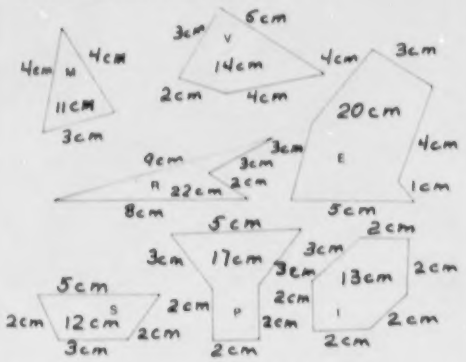
Point in turn to the mark for 4.2 cm, 4.6 cm, 5.8 cm, and 5.3 cm. Each time ask which is the nearest centimeter (4, 5, 6, and 5, respectively). Continue this activity, pointing to several other marks (but avoiding for the moment the marks for 4.5 cm and 5.5 cm), until you receive assurance of general understanding. Then point to the mark for 4.5 cm and ask which is the nearest centimeter. Some may say "4 cm" and others may say "5 cm". Elicit the observation that the mark is halfway between the two.

T: Let's make an agreement. If the endpoint of the line segment falls halfway between two centimeters marks, then we will use the larger. Using this agreement, what is the length to the nearest centimeter for a segment ending at 4.5 cm? (5 cm); at 5.5 cm? (6 cm); at 116.5 cm? (117 cm); at 2,762.5 cm? (2,763 cm) We use the same agreement if we are measuring to the nearest meter or decimeter or millimeter, or any other unit. To the nearest meter, what is the length of a segment ending at 27.4 m? (27 m); at 38.5 m? (39 m) To the nearest decimeter, what is the length of a segment ending at 46.5 dm? (47 dm)

Distribute Worksheets G01 and G02 in turn. Watch out for the common mistake of placing the mark at "1" instead of "0" or the end of the ruler at one end of a line segment being measured.

Name _____ G01 *

Measure the sides of each shape to the nearest centimeter.
Write the length of each side next to it.
Inside each shape, write the total of the lengths of its sides.



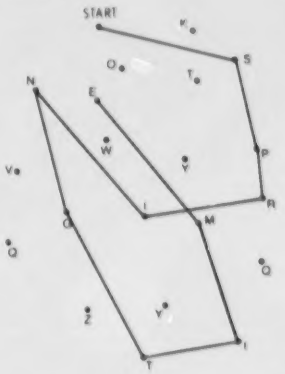
The letter of a shape is matched to the number inside the shape.
Use the matching to decode this message.

M	I	S	S	I	S	S	I	P	P	I		R	I	V	E	R
11	13	12	12	13	12	12	13	17	17	13		22	13	14	20	22

Name _____ G02 **

This is a treasure map. Start at the dot marked START and draw a zigzag.
Use the list of lengths for the segments of the zigzag. The second dot is
5.7 cm from the dot marked START. Write the letters from the dots in order
below to find the name of the treasure.

5.7 cm
3.8 cm
2.0 cm
4.9 cm
6.8 cm
5.1 cm
6.7 cm
3.9 cm
5.1 cm
6.5 cm



S P R I N G T I M E

3G

1.2 Area #1

This is the first of several lessons on area. You will need a grid board, geoboard, or grid paper (see the Note on Grids in the introduction to this chapter) to demonstrate calculating areas of regions by counting squares or parts of squares enclosed in a region.

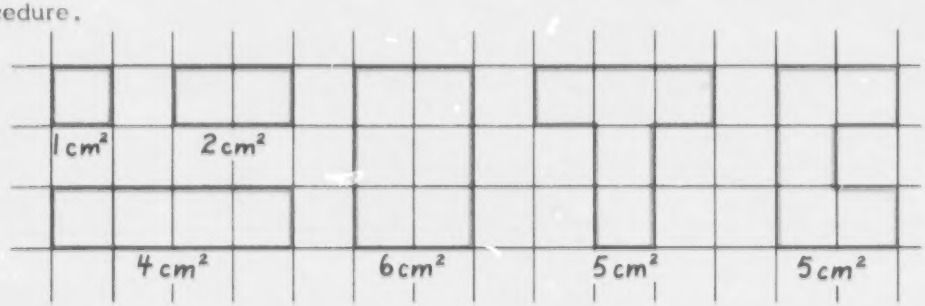
T: Today we will measure the area of shapes that we can draw on the grid board. We will pretend that the length of a side of one of the squares of the grid board is one centimeter. Then the area of a square on the grid board is one square centimeter.

Write this abbreviation on the board.

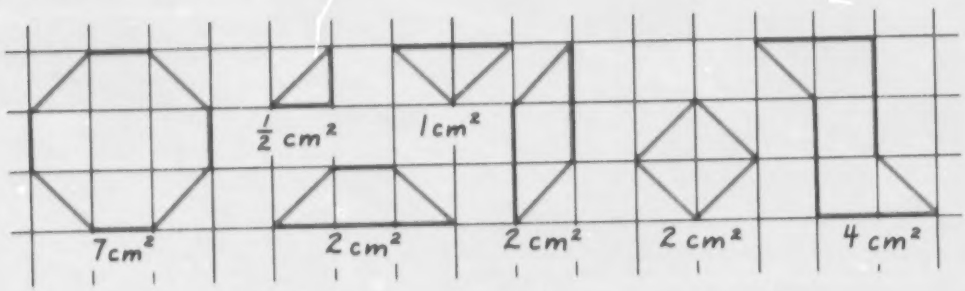
1 square centimeter = 1 cm^2

T: Now I will draw a shape on the grid board and I want you to find its area by counting the number of square centimeters it contains.

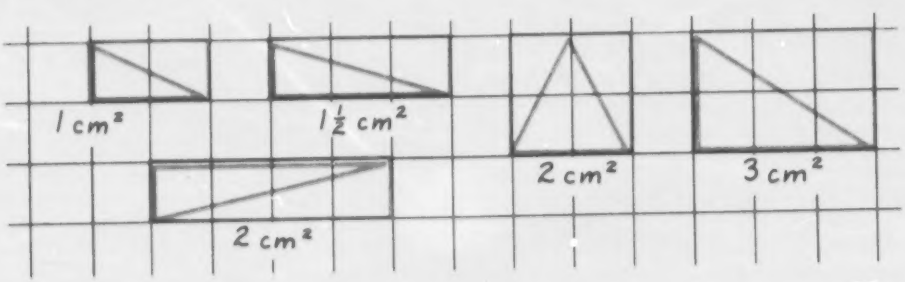
Draw some of the following shapes, one at a time. After drawing each shape, encourage discussion among the students to determine its area. If the grid board is large enough to allow the areas to be written neatly on or near the shapes, do so. Otherwise, make a list of the areas on the chalkboard. Do not forget to write the unit "cm^2" with the magnitude of each area. Draw enough of the shapes to ensure that most students recall the counting procedure.



Draw a selection of the shapes from the next illustration and lead the class in finding their areas. Start with the triangular half-square and emphasize that its area is one-half square centimeter. The other shapes can be decomposed into a collection of squares and triangular half-squares to find their areas.

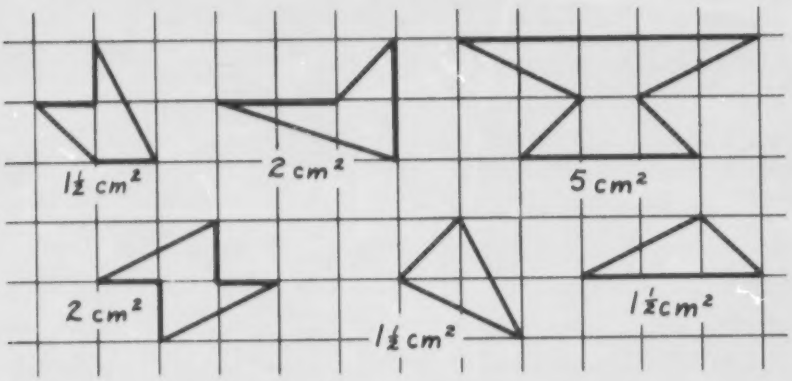


In turn, draw each of the red triangles in the following illustration and lead the class in finding their areas. A good method to emphasize is to enclose a triangle in a rectangle twice its size. The blue lines in the illustration show a rectangle enclosing each triangle. Many students try instead to count by matching small pieces to large pieces with a visual estimation of fractions — a practice to be gently discouraged.

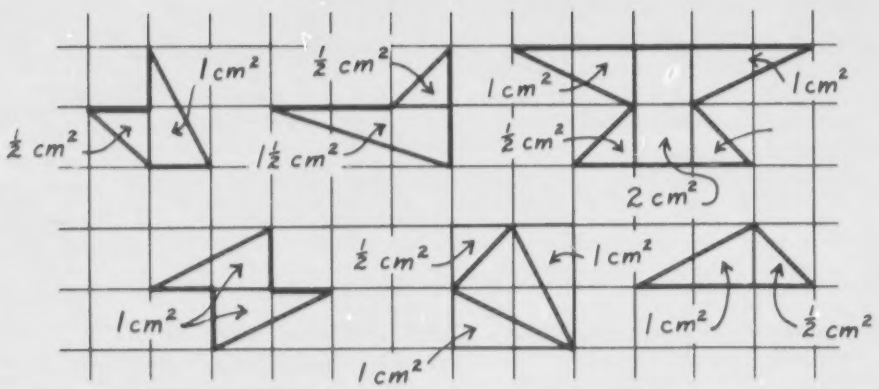


Present a selection of the shapes from the next illustration. Each time you present a shape, invite students to suggest methods for finding the area. Discuss their methods and let the class participate in deciding their validity. For these shapes, the decomposition method of finding areas is more complex. The technique to emphasize is that of finding constituent triangles and rectangles and then finding the areas of the individual pieces. For each shape, one possible decomposition is indicated with blue lines in the key which

follows. Present as many of these shapes as the level of interest in the class allows and then distribute Worksheets G03, G04, G05, and G06 in turn for individual work. Students will need colored pencils for the worksheets.



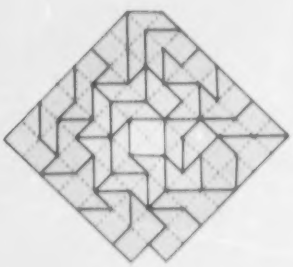
Solutions



Name _____ G 03 *

Color all shapes of area 1½ cm² with the same color.

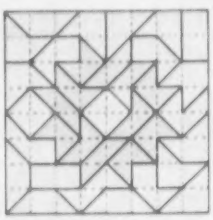
Use another color to color all shapes of area 2½ cm².



Name _____ G 04 **

Color all shapes of area 2 cm² the same color.

Use another color to color all shapes of area 1½ cm².

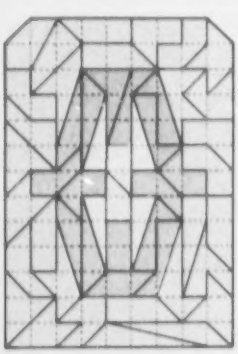


Name _____ G 05 ***

Use one color to color all shapes of area 1½ cm².

Use another color to color all shapes of area 2 cm².

Use a third color to color all shapes of area 2½ cm².



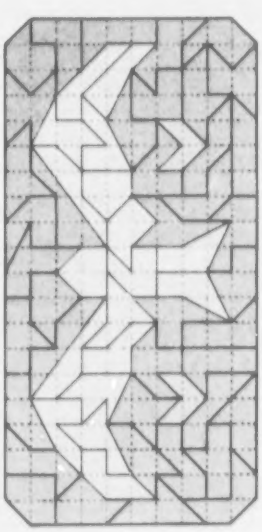
G 06 ****

Name _____

Use one color to color all shapes of area 2 cm².

Use another color to color all shapes of area 3 cm².

Use a third color to color all shapes of area 3½ cm².



182

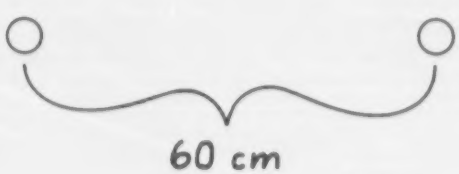
6 G

1.3 Length #2

This lesson provides more practice with the ruler in constructing line segments of prescribed length and zigzags under prescribed conditions. If necessary, review the use of the ruler and the measurement of the length of segments in centimeters and millimeters as in the beginning of Section 1.1. Each problem on the worksheets for this lesson involves constructing a zigzag connecting two or more circles, given the lengths of the pieces of the zigzag. The problem is to choose the lengths in an appropriate order and to place the pieces of the zigzag so that one end of the zigzag is in the starting circle and the other end is in the ending circle.

First, lead a collective solution of a similar problem. Draw two freehand circles on the board, each with a diameter of about 10 cm and with centers about 60 cm apart. List two lengths on the board next to your picture.

50 cm
40 cm



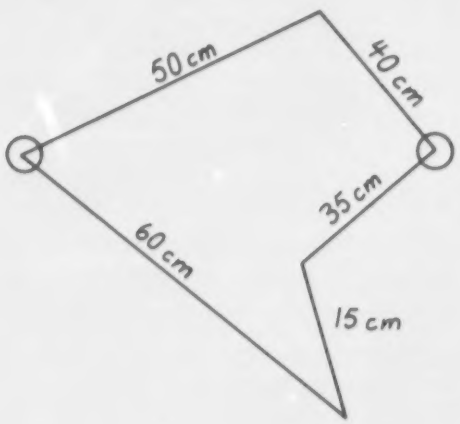
State the problem: draw a two piece zigzag starting anywhere inside one circle and ending inside the other circle. One piece of the zigzag must be 50 cm long and the other piece 40 cm long.

When this problem is solved, list three more lengths on the board.

15 cm, 60 cm, 35 cm

Then lead the class to construct a three piece zigzag connecting the two circles. (It is possible that the two zigzags will cross.)

50 cm
40 cm
---
15 cm
60 cm
35 cm



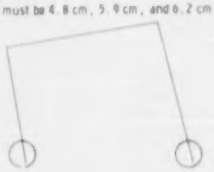
Distribute Worksheets G07, G08, and G09 in turn and monitor the individual work. Occasionally, discuss some of the students' solutions to the problems, or when many students have found solutions to the problems on Worksheet G07, you might discuss these problems collectively. Such a discussion could suggest techniques to those students who are having difficulties.

Name _____ G 07 *

Draw a two piece zigzag starting in one circle and ending in the other. Make one piece 5 cm long and the other 7 cm long.

Draw a three piece zigzag starting in one circle and ending in the other. The lengths of the pieces must be 2 cm, 4 cm, and 8 cm.

Draw a three piece zigzag starting in one circle and ending in the other. The lengths of the pieces must be 4.8 cm, 5.9 cm, and 6.2 cm.

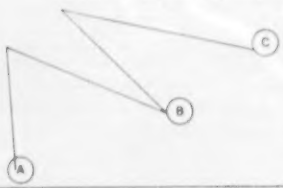


(Many solutions are possible.)

Name _____ G 08 **

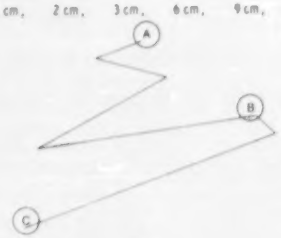
Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once:

5 cm, 6 cm, 7 cm, 8 cm.



Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once:

1 cm, 2 cm, 3 cm, 6 cm, 9 cm, 11 cm.

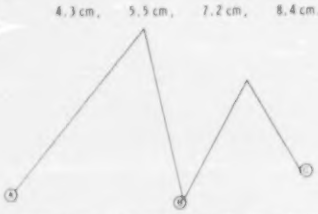


(Many solutions are possible.)

Name _____ G 09 ***

Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once:

4.3 cm, 5.5 cm, 7.2 cm, 8.4 cm.



Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once:

2.5 cm, 3.6 cm, 4.9 cm, 10.4 cm, 11.2 cm.

(Many solutions are possible.)

1.4 Area #2

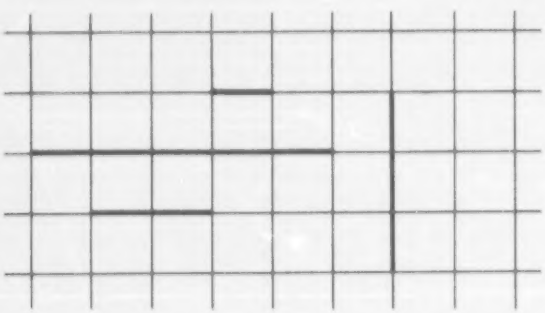
Exercise 1

Use a large demonstration grid board, geoboard, or grid paper (see the Note on Grids in the introduction to this chapter) to present line segments and rectangles to review finding the perimeter and area of a rectangle.

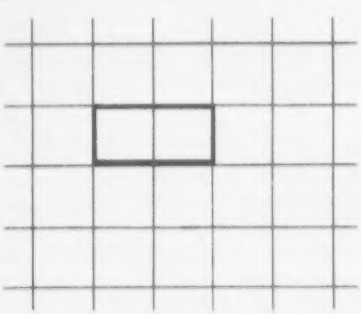
T: Let's pretend that the length of a side of a square on the grid is one meter.

Keep this unit designation of "1 m" on the board throughout Exercise 1 as a reminder. The lesson is presented here as if you will use a grid board, but a geoboard is preferable.

Draw a few line segments on the grid board as shown in the next illustration. As you draw each segment, ask a student to state the length of the segment.



Erase the line segments from the grid and draw a rectangle.

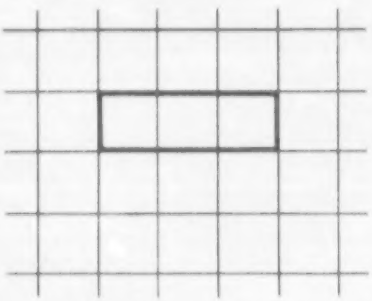


10 G

T: What is the perimeter of this rectangle?

S: 6 m.

Draw another rectangle on the grid.

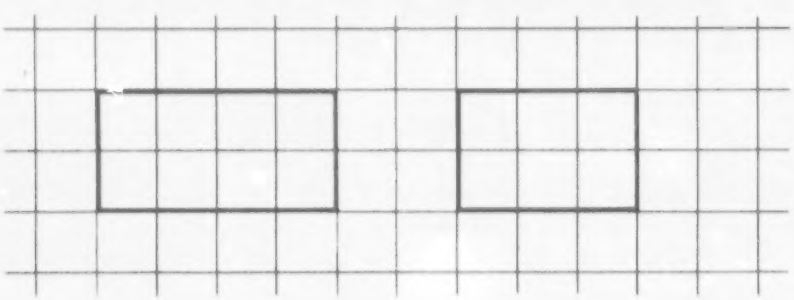


T: What is the perimeter of this rectangle?

S: 8 m.

If there is room on your grid, build a collection of rectangles on the grid board. Do not erase one as you proceed to draw another.

Continue with two more rectangles.



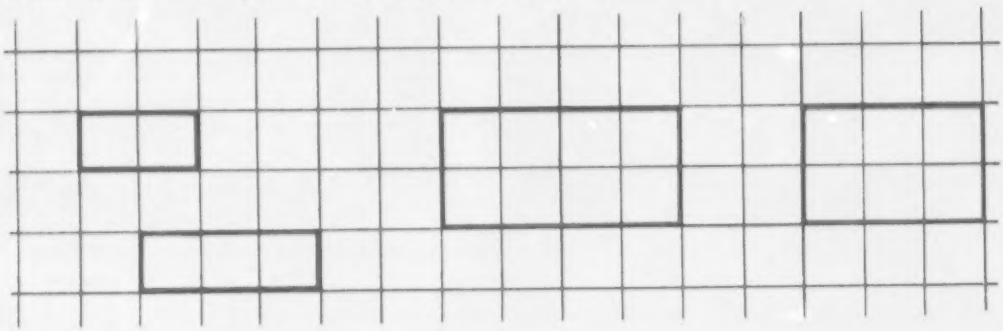
T: So far we have only been measuring the perimeter of the rectangles. Let's measure their areas as well. If the length of one side of a square of the grid is one meter, as I have written here, what is the area of one square?

S: One square meter.

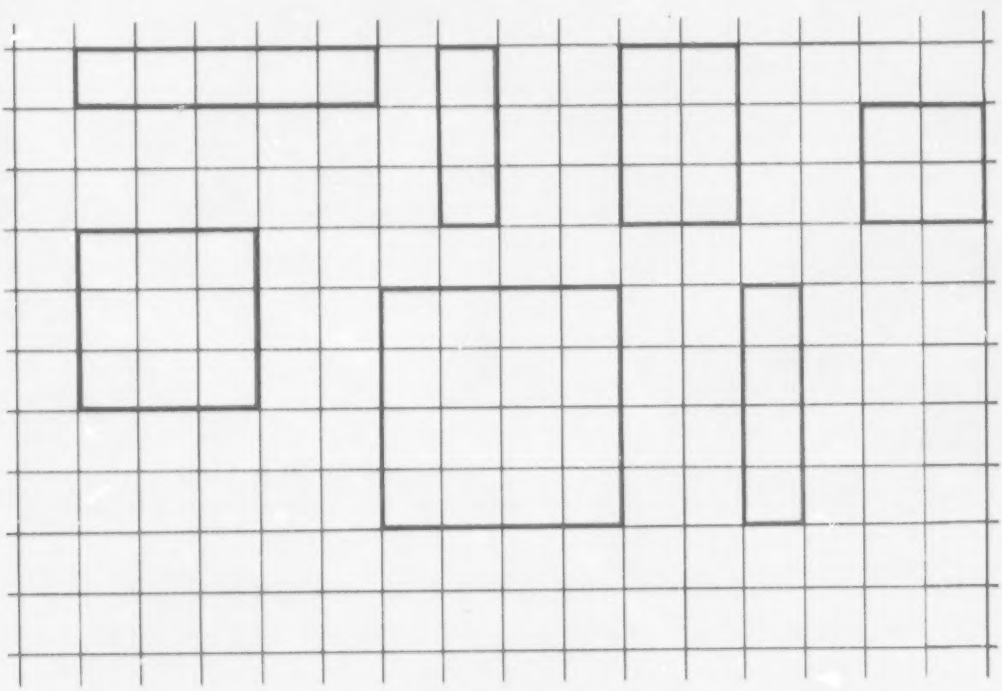
Write this abbreviation on the board next to the grid unit designation.

1 square meter = 1 m^2

Point to the rectangles on the grid (or redraw them in turn).



Ask for the area of each one. [Answers: 2 m^2, 3 m^2, 8 m^2, 6 m^2] Then draw a few more rectangles whose sides are aligned with the grid lines.



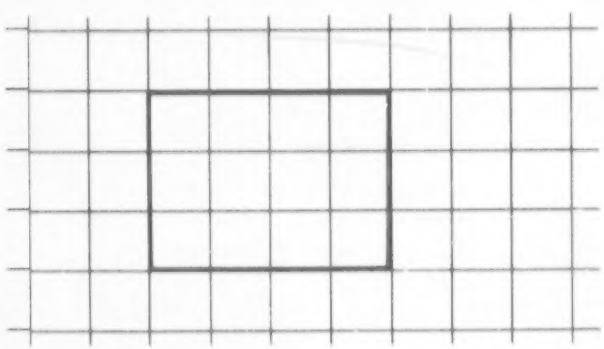
10 G

Some students will simply count one by one the squares of the grid which comprise the rectangles; some may use other counting methods. Encourage a multiplication method for calculating areas or methods which suggest multiplication.

T: Who has a way of finding the area of the rectangles?

S: I count the number of rows of squares in the rectangle and the number of squares in each row. I get the area by multiplying the two numbers.

T: Look at this rectangle. What is its area?



S: Three rows and four squares in each row. The area is 12.

T: 12 what?

S: 12 square meters.

Do not allow a student to state an area without providing the unit.

T: Correct! The length of one side of this rectangle is 3 meters and the length of the other side is 4 meters. The area is 3 x 4 square meters or 12 square meters. Let's check that method with the other rectangles.

Look at the other rectangles. For each one, ask a student to calculate the length and width, and then to perform the multiplication to find its area.

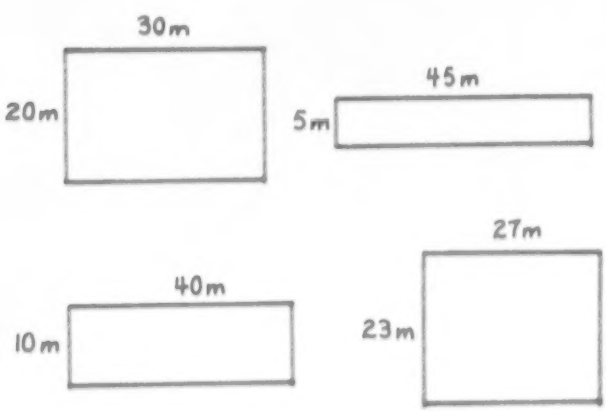
Exercise 2

T: I have a problem for you. A farmer has 100 meters of fencing material. The farmer wants to build a rectangular pen for cows and use all of the fencing material. (Draw a rectangle on the board to represent the pen.) What will the perimeter of the pen be?

S: Using all of 100 meters of fencing will make a perimeter of 100 meters.

T: What size could such a pen be?

Of course, there are many correct answers. Draw a small scale rectangle on the chalkboard for each answer as it is suggested. Draw one pair of opposite sides in blue and the other in red as illustrated below. Be sure to check that the perimeter of each suggested rectangle is 100 m. Begin a chart as shown below to record the answers. After a few solutions are recorded, change to asking only for a length for the blue from one student and then for the corresponding red from another.



● ●	● ●
20 m	30 m
10 m	40 m
5 m	45 m
23 m	27 m
35 m	15 m
32.5 m	17.5 m

10 G

Eventually most students will recognize that the sum of the lengths of two adjacent sides of a correct solution is 50 m.

T: Now let's find the area of these rectangles.

Pick one of your listed solutions which has both dimensions a multiple of 10, say 20 m by 30 m.

T: What is the area of this rectangle?

S: 20 x 30 = 600, so the area is 600 square meters.

Refer to the corresponding drawing on your chalkboard to illustrate the calculation. Extend the chart to list the areas of the rectangles. If necessary, demonstrate some of the multiplications at the board.

● ●	● ●	AREA
20 m	30 m	600 m^2
10 m	40 m	400 m^2
5 m	45 m	225 m^2
23 m	27 m	621 m^2
35 m	15 m	525 m^2
32.5 m	17.5 m	568.75 m^2

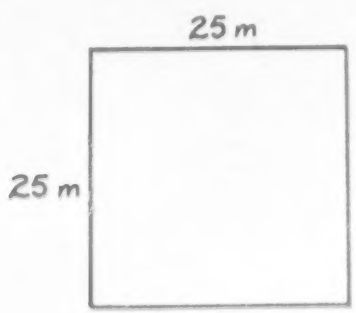
T: Our farmer is very thrifty and would like to build a pen with the largest possible area. Which of these rectangles has the largest area? I have a worksheet for you to list more solutions to the farmer's problem. Try to find a solution with the largest possible area.

Distribute Worksheet G10. Monitor the students' work on the problem and, occasionally, when you observe an interesting solution (e.g., one involving non-integer decimals or one with a large area close to the maximum), add it to the chart. The maximum area is that of a rectangle whose dimensions are 25 m by 25 m, that is, 625 square meters. Be sure that this solution gets recorded in your chart at some time.

This is an excellent opportunity for students to use hand-calculators. Use of this tool allows students to consider many solutions and to concentrate on the problem (to find a solution with largest area) rather than on calculation. Moreover, the student with a hand-calculator may be encouraged to explore esoteric solutions involving decimals.

Depending on the interest and sophistication of your students, present the following optional justification for the best possible solution being a square.

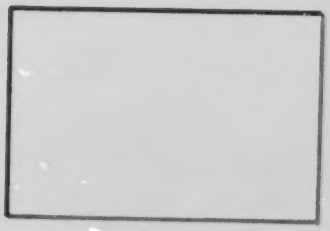
Use white (printed in black on this page) and red chalk to draw a square on the board, and denote the length of its sides as "25 m".



T: Here is a picture of the farmer's pen built as a square. Suppose we have another solution, like this.

Use white and blue chalk to draw another solution next to the square. Be sure to have the scale of the rectangle match that of the square, that is, the perimeter of the rectangle should be roughly equal to that of the square.

10 G

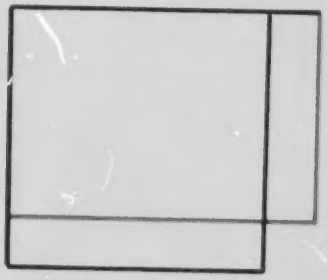


T: How can we compare the areas of the square and the rectangle?

Encourage discussion of the question and then present a suggestion as follows.

T: We could put one on top of the other and compare them. I will draw another rectangle the same size on top of the square.

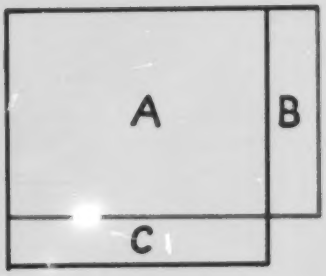
Draw a rectangle on top of the square as shown below. Be careful to make it as close in size to the first rectangle you drew as possible. Then erase the first rectangle you drew.



10 G

T: A large part of the square is covered by the rectangle. But part of the square is not covered, and part of the rectangle does not cover any of the square. Let's name the three parts: A, B, and C.

Label your drawing.



T: Now the square has two parts, A and C, and the rectangle has two parts, A and B. Which has the larger area, B or C?

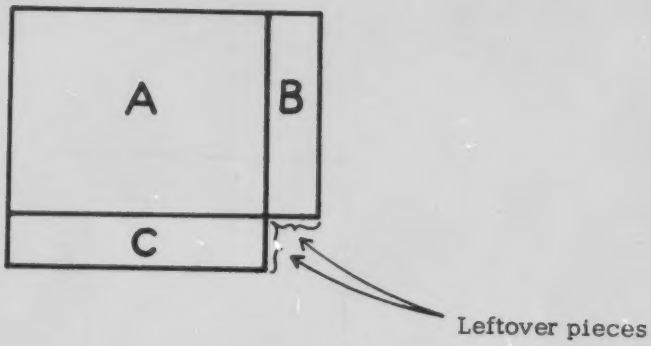
S: I think B is bigger.

S: I think that C is bigger.

T: Let's see if we can find out who is correct. This blue segment (trace the vertical blue segment) is exactly the same length as part of this red segment (trace the left-hand side of rectangle B). This red segment (trace the horizontal red segment) is exactly the same length as this blue segment (trace the top side of rectangle C). Now we have matched pieces of blue segments with pieces of the red segments, and we have the two left-over pieces.

10G

Use two hands to simultaneously trace matched pieces of blue and of red, and then point to the leftover pieces.



NOTE: The object of the next section of the discussion is to observe that the leftover pieces have the same length.

T: What is the total length of the blue segments?

S: 50 m, one-half of the perimeter of the rectangle.

T: And what is the total length of the red segments?

S: Also 50 m, one-half of the perimeter of the square.

T: When we take the matched pieces away from the blue, one piece is leftover.

Trace the relevant pieces and point to the leftover blue piece.

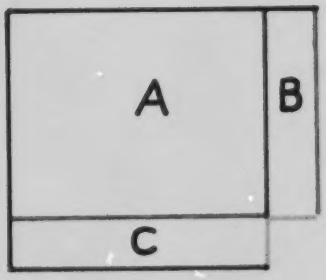
T: And when we take the matched pieces away from the red, one piece is leftover.

Trace the relevant pieces and point to the leftover red piece.

S: Both leftovers are the same length.

T: Correct! Let's make each of the leftover pieces yellow.

Erase the leftover pieces and replace them with yellow segments.



T: What is the area of rectangle B?

S: Yellow times (vertical) blue.

Trace the yellow and the blue sides of B.

T: And what is the area of rectangle C?

S: Yellow times (horizontal) red.

Trace the yellow and the red sides of C.

T: Then which is larger, rectangle B or rectangle C?

S: C is larger than B, because the red side of C is longer than the blue side of B.

10 G

Verify this assertion by tracing the appropriate segments.

T: Then which is larger, A together with B or A together with C?

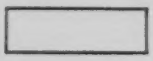
S: A together with C is larger, because C is larger than B.

T: So the square is a better solution for the farmer than the rectangle.

Name _____ G 10 *

What is the largest possible area ?

perimeter
100 m



● ●	● ●	Area
20 m	30 m	600 m^2
5m	45m	225m^2
16m	34m	544m^2
1m	49m	49m^2
18.3m	31.7m	580.11m^2
25m	25m	625m^2
40m	10m	400m^2

Use the other side to record other solutions.

(Many solutions are possible.)

1.5 Area #3

Display a grid board (or geoboard). Start with a unit (side of a square of the grid) designation of "1 dm" and periodically change it among "1 m", "1 dm", and "1 cm". Draw a variety of rectangles and ask students to calculate their areas and perimeters using the appropriate units. Encourage the technique of multiplying the lengths of two adjacent sides to find the area.

Recall the problem of the farmer with 100 meters of fence who wanted to build a pen with largest possible area.

Draw a representative rectangle.

perimeter = 100 m

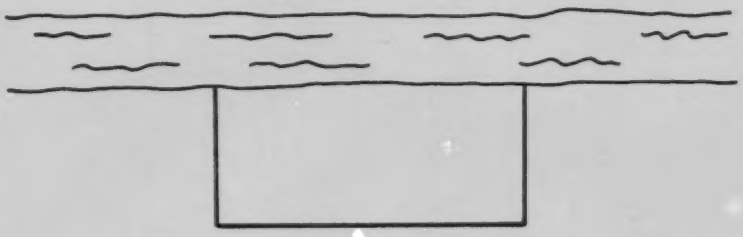
Solicit solutions and begin a chart as in the last lesson.

● ●	● ●	AREA
20 m	30 m	600 m^2
5 m	45 m	225 m^2
25 m	25 m	625 m^2
24.5 m	25.5 m	624.75 m^2
26 m	24 m	624 m^2
20.25 m	29.75 m	602.4375 m^2

Continue until six or eight entries are in the chart, including the solution for largest area, 25 m by 25 m. The class should remember that a square pen has the largest possible area.

T: There is another farmer in the area, who has a stream flowing through the farm. This farmer also has 100 meters of fencing material and wants to build a rectangular pen so that the stream forms one side.

Draw an illustration on the board.



T: What are some solutions to this farmer's problem?

Begin a chart to record the answers. For each suggestion, check that the total length of the three fence sides of the rectangle is 100 m. In order to engage more students in the mental calculations, sometimes accept a length for the blue (or red) segments from one student and then ask another student to calculate the necessary length of the other side(s).

● ●	●
20 m	60 m
15 m	70 m
40 m	20 m
5 m	90 m

11G

After several solutions have been given, ask students to calculate the areas as well. Continue until at least six or eight solutions have been recorded in your chart.

● ●	●	AREA
20 m	60 m	1,200 m^2
15 m	70 m	1,050 m^2
40 m	20 m	800 m^2
5 m	90 m	450 m^2
32.5 m	35 m	1,137.5 m^2
27.25 m	45.5 m	1,239.875 m^2

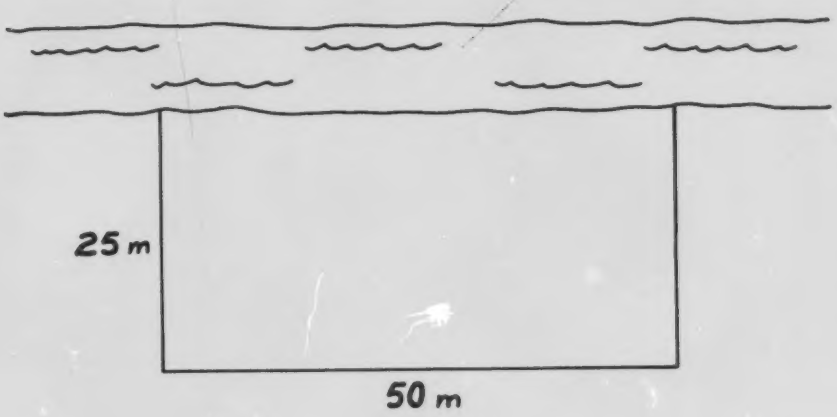
T: This farmer is also thrifty and wants to build a pen with the largest possible area. I have a worksheet for you to investigate the problem.

Distribute Worksheet G11. This is another excellent opportunity to use hand-calculators in order to focus attention on the problem of maximizing the area of the pen, rather than on the calculations. With hand-calculators available, you can encourage students to discover esoteric solutions such as 26.37 m (blue) by 47.26 m (red), which yields an area of 1,246.2462 m^2. The maximum area to be had is 1,250 m^2, with a rectangle of size 25 m by 50 m.

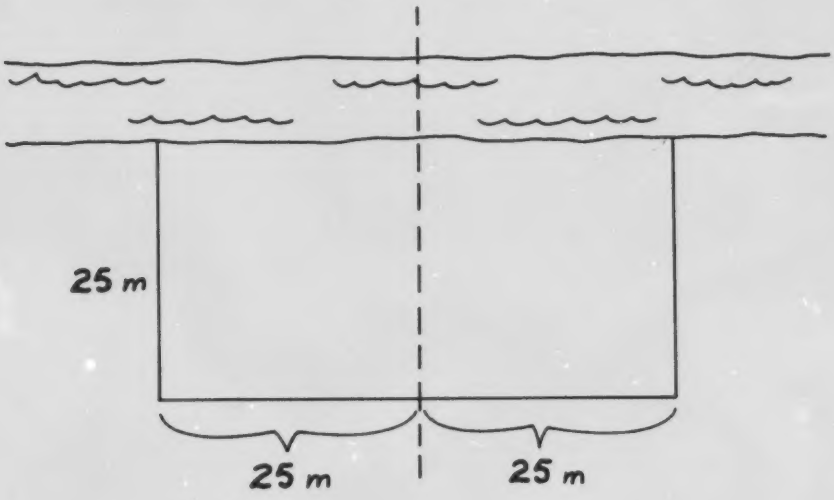
As students find the solution with largest area and are convinced of its correctness, ask them to repeat the exercise with other lengths of fencing, say 200 m, 300 m, or 500 m. The maximum areas in these cases are 5,000 m^2, 11,250 m^2, and 31,250 m^2, respectively. In each case, the best solution is a rectangular pen using half of the fencing for the red side and distributing the remaining fencing equally between the blue sides.

If the interest and abilities of your class permit, complete the lesson with a justification of the best possible solution, 50 m by 25 m. This option is similar to that suggested at the end of the lesson of Section 1.4. An outline is given here. As in Section 1.4, the discussion should be informal.

Begin by drawing a rectangle in the proportion of 25 m by 50 m against a stream.

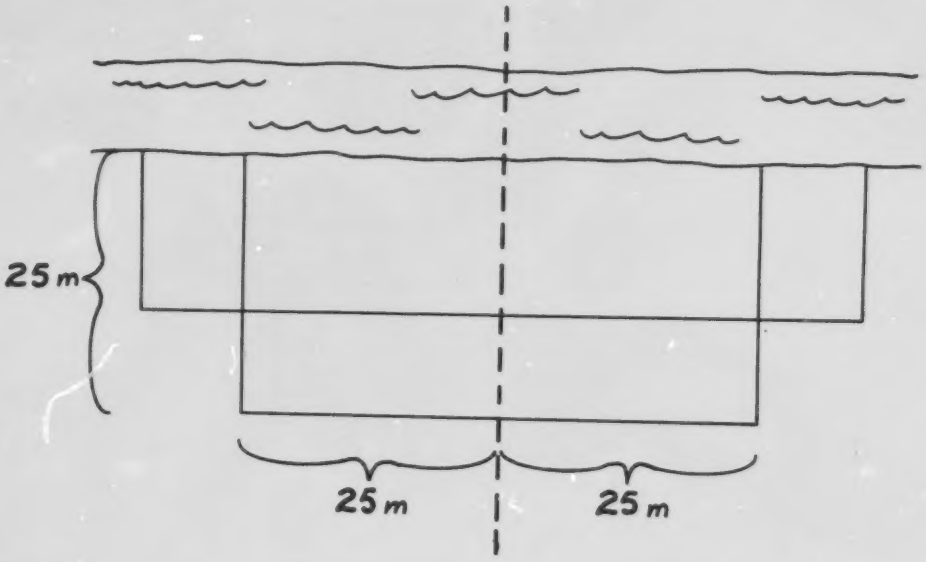


Then divide the rectangle by a vertical line into two equal pieces.

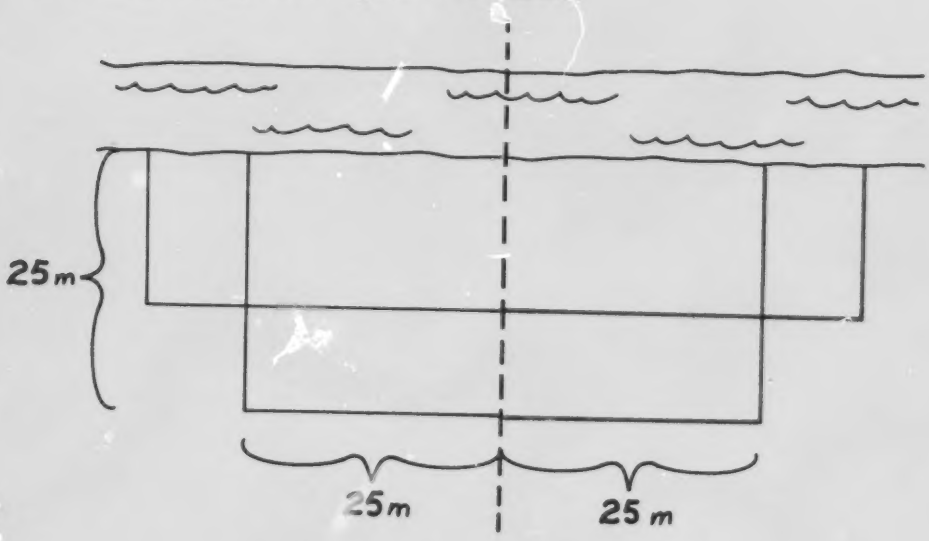


11 G

Now draw another solution rectangle oriented on top of the first as shown below. The dotted dividing line should also divide the second rectangle into two equal pieces.



Then color red and blue segments as shown.

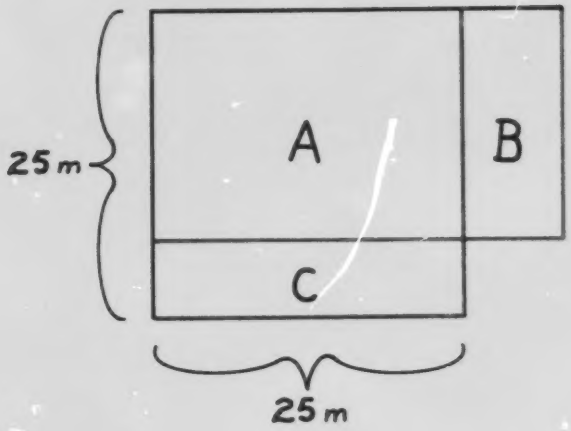


Initiate the discussion as you draw the rectangles. Make two observations in the discussion:

1) The dotted dividing line divides each solution rectangle in half. The pen with the larger half is the larger pen. So, the problem is to compare the areas of the two half pens.

2) The total length of the red segments is 50 m, and the total length of the blue segments is also 50 m. Each is one-half of the total length of fencing or one-half of 100 m.

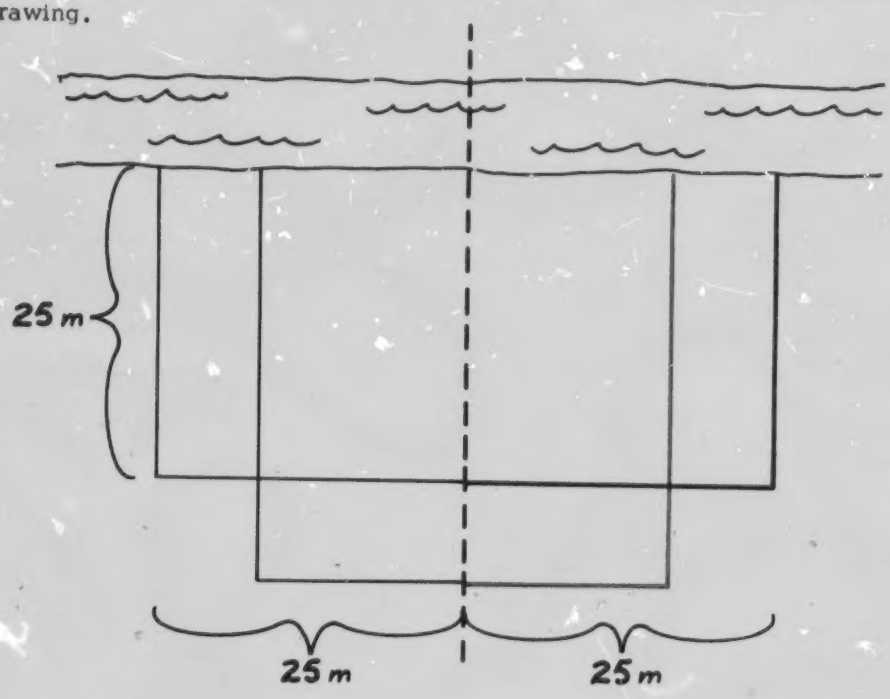
Now draw another copy of the right half of the illustration and label the constituent rectangles.



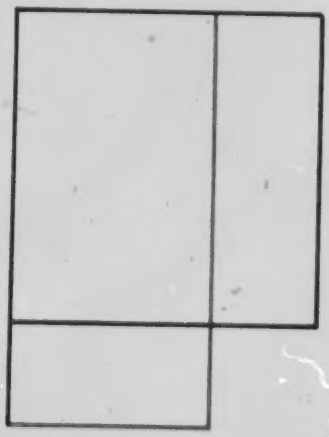
Someone may remember that this is the same drawing as that for the lesson of Section 1.4. The argument given there showed that A and C together have a greater area than A and B together, because C is bigger than B. Review as much of that argument as you think desirable. Return to the original drawing to conclude that the 25 m by 50 m rectangle has a larger area than the other rectangle.

11 G

NOTE: In order to complete the argument, another case remains to be considered. The blue and red rectangles could be oriented as in the following drawing.



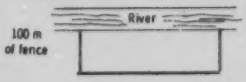
Then, extracting the right-hand side gives



which is similar to the drawing analyzed before. So it is sufficient to have considered only the first case.

Name _____ G 11 *

What is the largest possible area ?



● ●	●	Area
20 m	60 m	1200 m^2
10 m	80 m	800 m^2
1 m	98 m	98 m^2
15 m	70 m	1050 m^2
25 m	50 m	1250 m^2
38.5 m	23 m	885.5 m^2
48 m	4 m	192 m^2

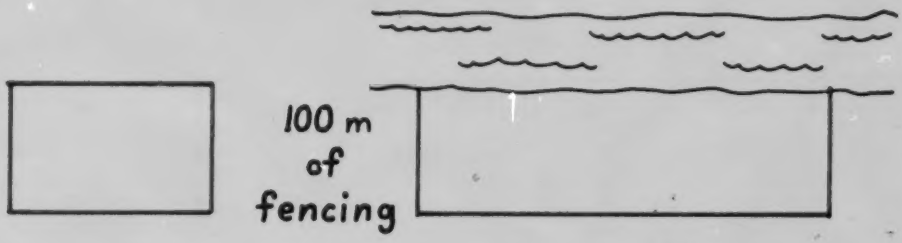
Use the other side to record other solutions.

(Many solutions are possible.)

12G

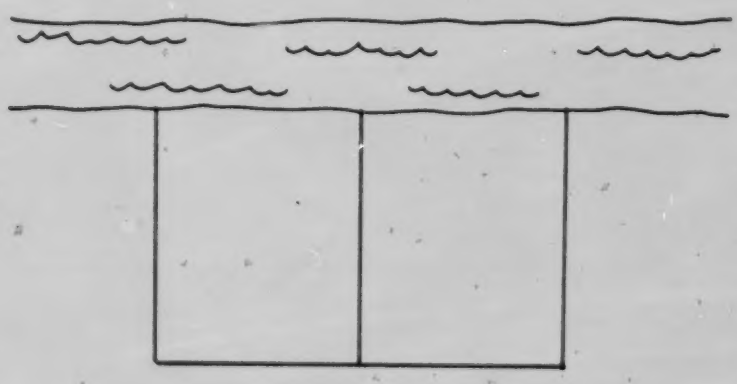
1.6 Area #4

Recall the problem of the two farmers from the previous two lessons. Each farmer had 100 m of fencing to construct a rectangular pen, but the second farmer had a stream to use for one side of the pen. Each wanted to build a pen of largest possible area. Draw illustrative rectangles as you recall the two situations.



T: Downstream from the second farmer there is another farmer who keeps sheep and goats. This farmer also has 100 m of fencing, but needs two pens.

Draw an illustration.



Ask for solutions and begin a chart as below. Again accept a suggestion from one student for the length of one blue segment or for the red segment and ask another student to calculate the other length.

● ● ●	●
30 m	10 m
15 m	55 m
21 m	37 m
$3\frac{1}{3}$ m	90 m
10.5 m	68.5 m

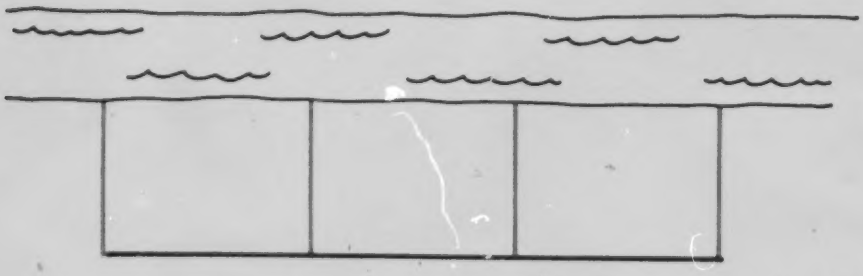
After a few solutions are recorded and checked, start the calculation of the areas of the rectangles.

● ● ●	●	TOTAL AREA
30 m	10 m	300 m^2
15 m	55 m	825 m^2
21 m	37 m	777 m^2
$3\frac{1}{3}$ m	90 m	300 m^2
10.5 m	68.5 m	719.25 m^2

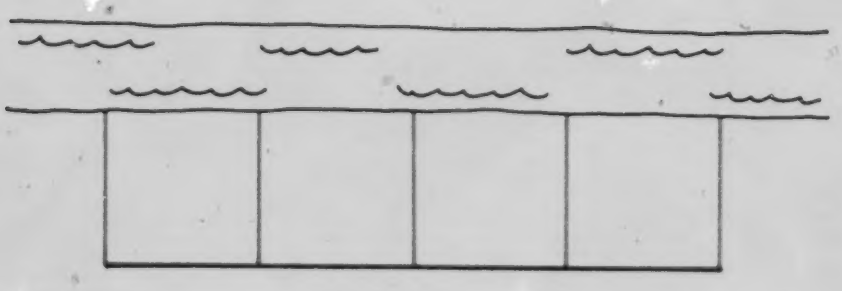
T: This is another thrifty farmer who wants to build pens to have the largest possible area. I have a worksheet for you to investigate the problem.

12 G

Distribute Worksheet G12. Again provide hand-calculators, if they are available. This time the maximum area is achieved by a rectangle of size $16\frac{2}{3}$ m by 50 m. If anyone discovers this solution and becomes convinced of its correctness, suggest a similar problem with three pens, then four pens, and so on. The best solution for three pens is 12.5 m by 50 m and for four pens is 10 m by 50 m.



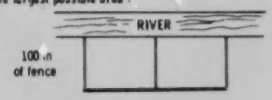
Three Pens



Four Pens

Name _____ G12 *

What is the largest possible area?



● ● ●	●	TOTAL AREA
15m	55m	825m²
20m	40m	800m²
1m	97m	97m²
18m	46m	828m²
17m	49m	833m²
16m	52m	832m²
16.5m	50.5m	833.25m²

Use the other side to record other solutions.

(Many solutions are possible.)

2

PARALLELISM

Suggestions of parallel lines are everywhere: the painted guide lines on a super highway, a parade of telephone poles, the opposite sides of a desk, the ruled lines on notebook paper. The architectural embodiments of parallelism indicate its role even in the earliest geometry. From its formal debut as a central concept of Euclid's Elements, a rich and fascinating study of parallel lines and related objects has developed over many centuries.

Although traditionally postponed to secondary school, the notions of parallelism appeal directly to one's intuition. Equipped with a rolling straightedge called a "translator" and a compelling dynamic definition of parallel line segments, students of the lessons in this chapter will perform free and open, but accurate constructions based on parallel line segments and parallelograms. In the lessons of CSMP Mathematics for the Intermediate Grades, Part II, students developed their familiarity with the translator in constructing families of parallel line segments. Those lessons introduced the basic contructions of the parallelogram and the parallelogram train, and then used these constructions to define the equipollence relation on couples of points.

A couple is simply a pair of points, one designated as first and the other second. In drawings, the ordering of a couple is shown by a colored arrow starting at the first point of the couple and ending at the second point. By drawing the arrows as (directed) line segments, we may use them as sides of parallelograms. Couples are equipollent if their arrows can be joined by a series of parallelograms with the arrows among successive opposite sides of the parallelograms. In addition, the sense of the arrows must all be the same; that is, if the starting points of two of the arrows are joined by a line segment and the ending points of the arrows are joined by another line segment, then the line segments together with the arrows form a parallelogram. (Colinear arrows are a separate case.) In the next illustration there are five equipollent couples shown with red arrows and three equipollent couples shown with blue arrows.

Time spent in reading the seven lessons on parallelism in Part II would be time well spent. If you have not taught those lessons, work through the problems in those lessons yourself.

The translator and parallelism are reintroduced to the students in two lessons on parallel projection. Constructions of couples and equipollent families of couples reappear in the third lesson, a prelude to three lessons on translations. A translation is a relation defined by equipollent families of couples. Both parallel projections and translations provide examples of functions and, therefore, utilize and supplement the study of functions in the Language and World of Numbers strands. The last lesson reviews the translator construction of a number line introduced in Part II. The construction is then generalized to the construction of a two-dimensional lattice. The lattice will be a useful tool for a variety of numerical problems to be introduced in CSMP Mathematics for the Intermediate Grades, Part IV.

The construction is easier to perform than to describe. The steps are illustrated below and keyed to this description.

A) First, draw a line segment ①.

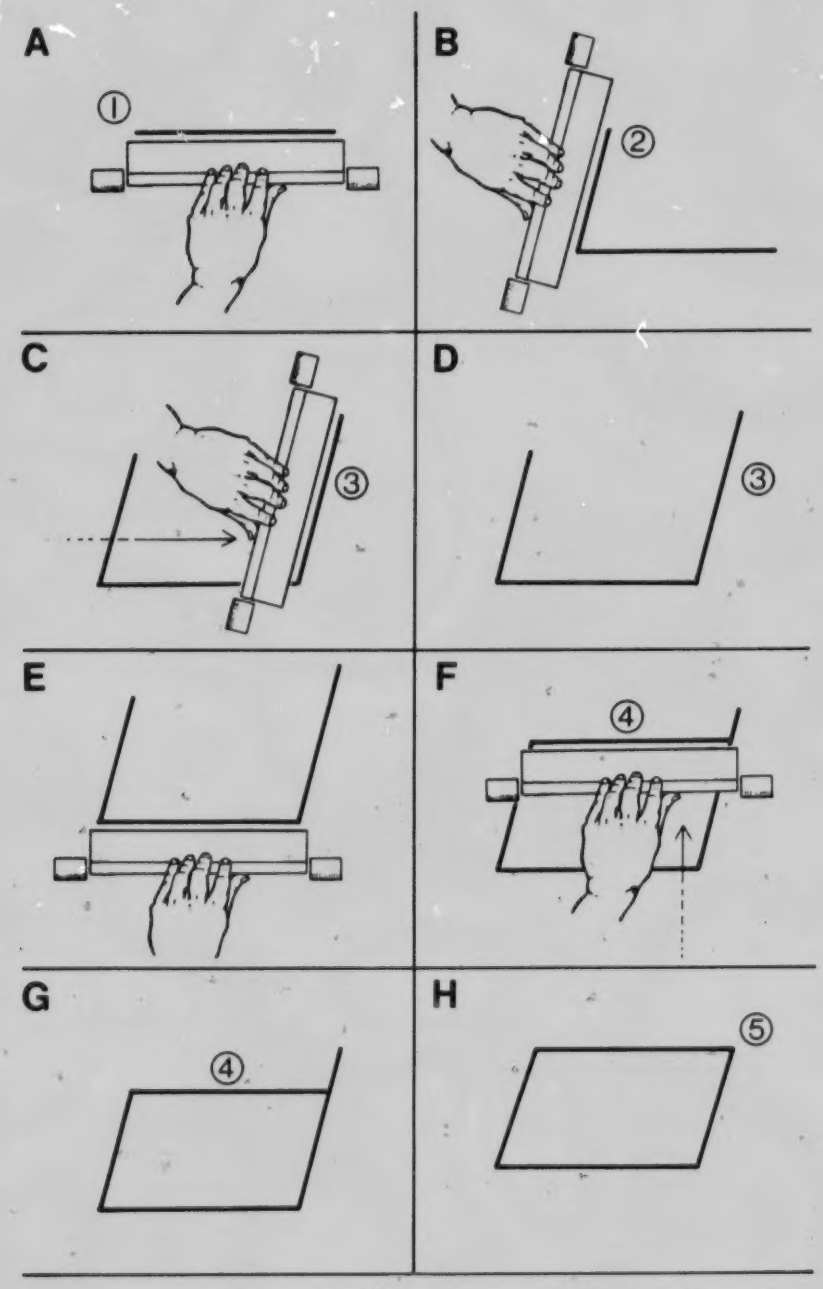
B) Then draw another line segment ② starting at one of the endpoints.

C) and D) Align the translator with the line segment ② and roll to the other end of the first line segment and draw another line segment ③.

E) Align the translator with the first line segment ①.

F) and G) Roll the translator to the endpoint of the second line segment ② and draw the last segment ④.

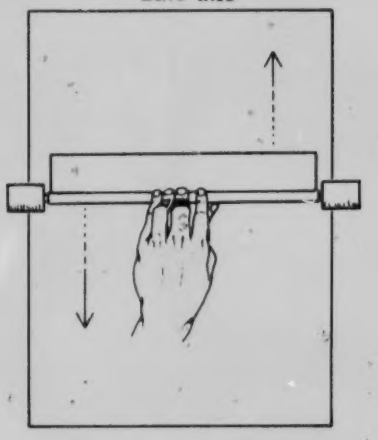
H) Tidy the corners by erasing any pieces of lines which extend beyond the corners ⑤.



NOTE ON TRANSLATORS

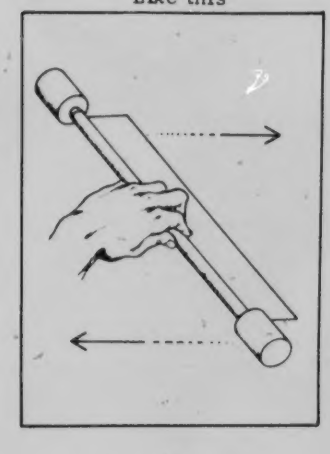
Before using the translator in the classroom, practice using it yourself. Take some time to practice using a student translator as well as the demonstration translator. Occasionally a construction at the board requires an odd posture. In introducing the class to the translator, mention the sharp corners of the metal blade as a warning against misuse. Provide a few small pieces of masking tape to each student so that paper can be anchored to the desk. This precaution will eliminate frustrating slipping of the paper while a student is drawing. Instruct the class in the proper manipulation of the translator. One should hold the plastic tube at the center and gently push or pull to move it across the paper or desk. The following pictures illustrate proper and improper handling.

Like this

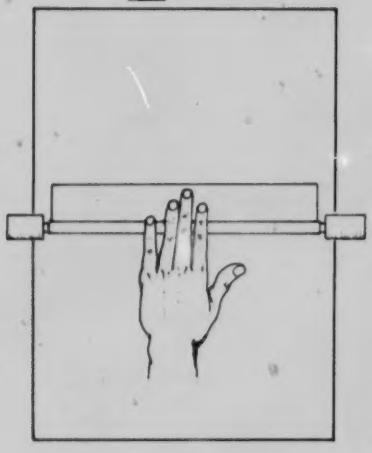


or

Like this

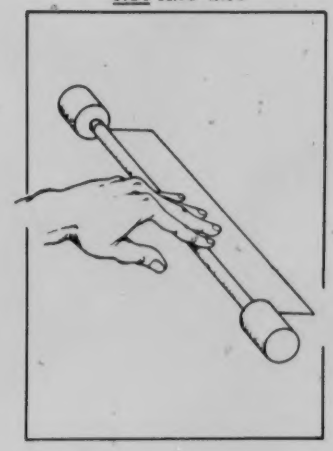


Not like this



and

Not like this



That the blade is not rigidly attached to the plastic tube should cause no problem when a slow steady pressure is used in rolling the translator. The advantage of this means of attachment is that blades may be easily repaired in the event of breakage. For repairs, simply provide yourself with a roll of strapping tape as used in the construction of the translator. However, we anticipate that repairs will only rarely be necessary.

CAPSULE LESSON SUMMARIES

Section 2.1: A falling bali motivates parallel projections. Project several sets of points. Introduce and play Follow My Finger. (Worksheets G13, G14)

Section 2.2: Project line segments and other curves. Play Follow My Finger. (Worksheets G15, G16, G17, G18)

Section 2.3: Review the construction of equipollent families of couples. Play Follow My Finger with equipollent couples.

Section 2.4: An equipollent family of couples determines a function called a translation. Find the images of several zigzags via translations determined by couples. (Worksheets G19, G20, G21, G22, G23)

Section 2.5: Find the images of letters of the alphabet via translations determined by couples.

Section 2.6: Find the images of several two-dimensional shapes via translations determined by couples. (Worksheets G24, G25, G26)

Section 2.7: Review the number line construction with the translator. Extend this construction to build a colorful two-dimensional lattice.

2G

2.1 Projection #1

NOTE: For this lesson, procure a small rubber ball with a good bounce.

Exercise 1

Begin the lesson by bouncing a ball on the floor from shoulder height. Each time, let it bounce only once before you catch it. Continue dropping and catching the ball occasionally through the ensuing discussion of the situation.

T: Think about the path of the ball as it leaves my hand until it first hits the floor.

S: The ball goes straight down.

T: Where does the ball hit the floor?

S: Directly under your hand.

T: Think about the path of the ball and think about how it is related to the wall.

S: They are both straight.

S: The ball falls and bounces up in a line parallel to the wall.

When an observation similar to the last remark is made, draw an illustration on the board.

T: Let's draw a line segment in this picture to show the path of the ball and the place where it hits the floor.

Invite the students to trace the path with a finger until someone has accurately traced it.

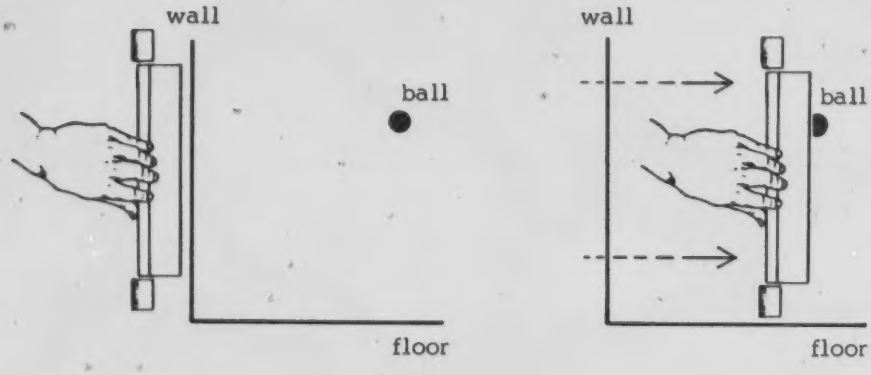
T: How can we draw this line segment accurately to show the ball's path?

S: Use a translator.

T: Tell me what to do.

S: Align the translator with the wall and roll it to the ball.

Repeat the instructions as you perform the actions.



S: Now draw a line segment from the ball to the floor.

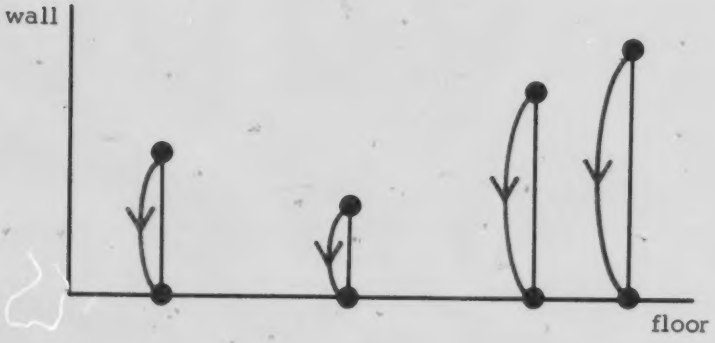
S: The ball will hit the floor where the line crosses the floor.

Draw a dot at the intersection of the floor line and the path line.

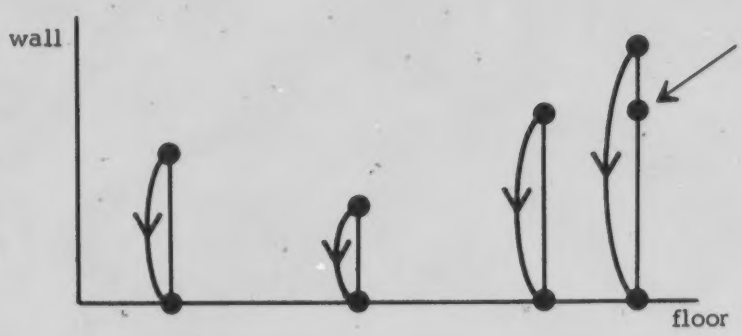
2G

T: Let's draw a red arrow from the ball to the point where it hits the floor.

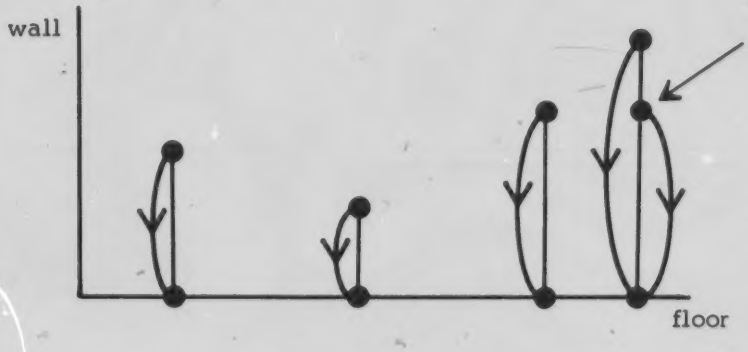
Then draw a number of other dots for balls in the picture. For each one, ask a student to trace its path to the floor and then use the translator to draw the path line accurately. Put a dot where the path line crosses the floor line to show the place where the ball hits the floor, and finish with a red arrow from the ball to the point where it hits the floor. Your drawing may look like the following.



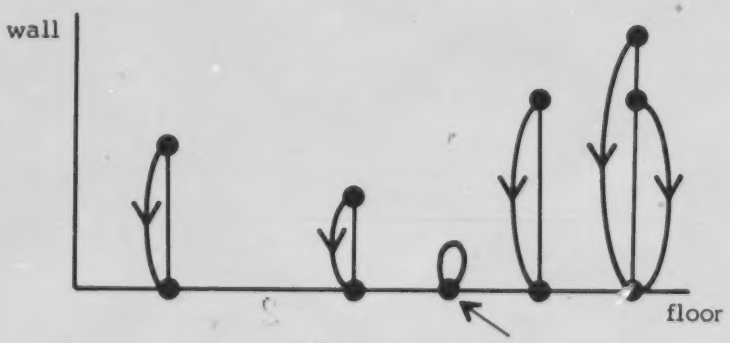
Next provide several special situations. First, draw a dot for a ball on one of the path lines already drawn.



Elicit discussion to observe that no new line need by drawn, and that the image point is already determined. Draw an arrow.

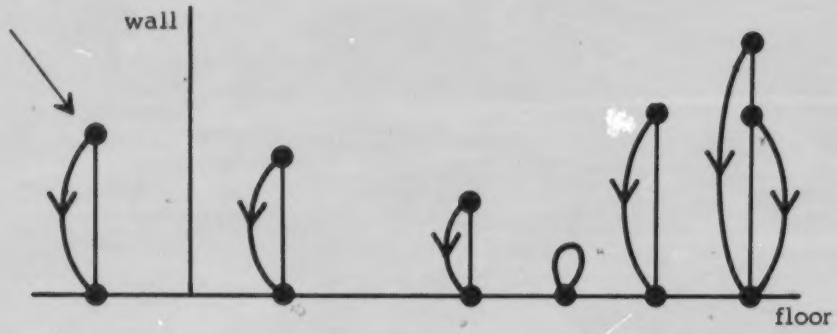


Second, draw a dot for a ball on the floor line. Elicit discussion to show that the ball does not fall, that no path line need be drawn, and that a red loop describes the situation.

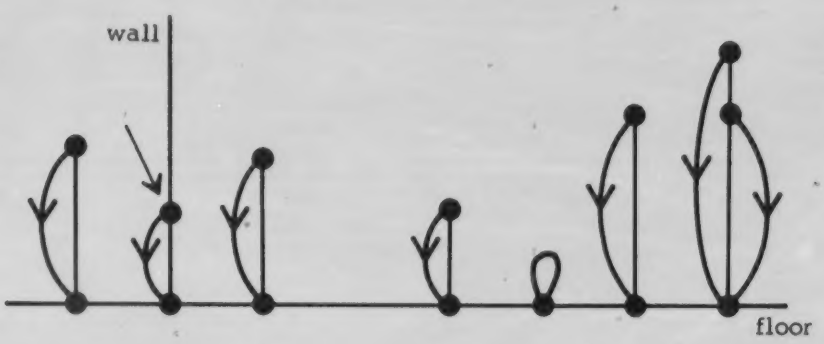


2G

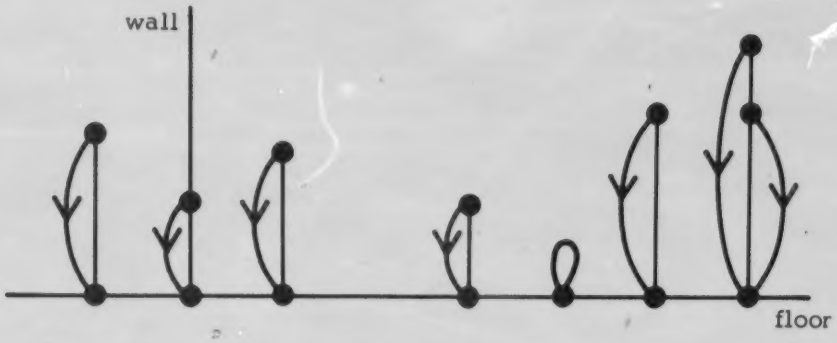
Third, draw a dot for a ball on the other side of the wall (left side in our illustration) and elicit discussion to observe that the floor line must be extended, but that the rest of the construction is as before.



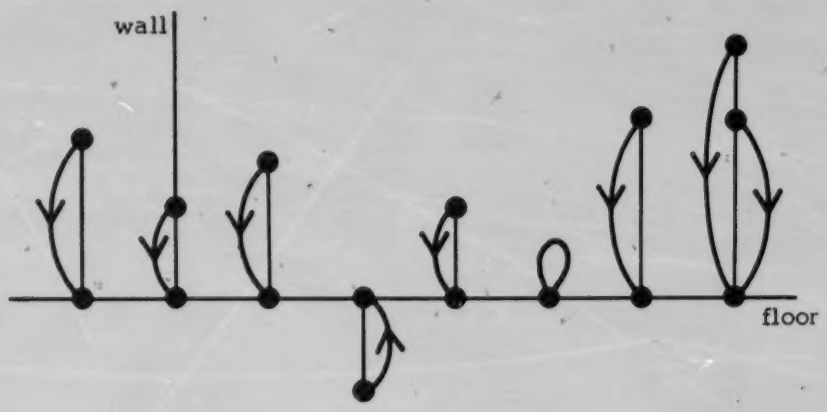
Fourth, draw a dot for a ball on the wall line and elicit the observation that the image point is already determined. Add an arrow.



Finally, draw a dot <u>below</u> the floor.



Students may observe that a ball would fall down and not up to the floor. Respond that instead of a ball this dot represents a gas-filled balloon. The problem is the same, where will the balloon hit the floor line? The technique of the construction is the same.



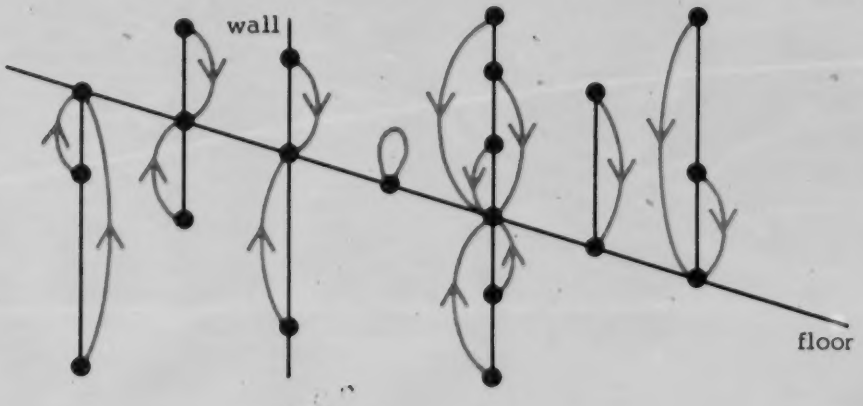
T: This is an example of PARALLEL PROJECTION. The balls are projected onto the floor parallel to the wall.

Write the words "parallel projection" and the underlined phrase above on the board and have the students read it a few times.

Exercise 2

T: Suppose we have a slanted floor.

Repeat Exercise 1 using a drawing similar to the following.



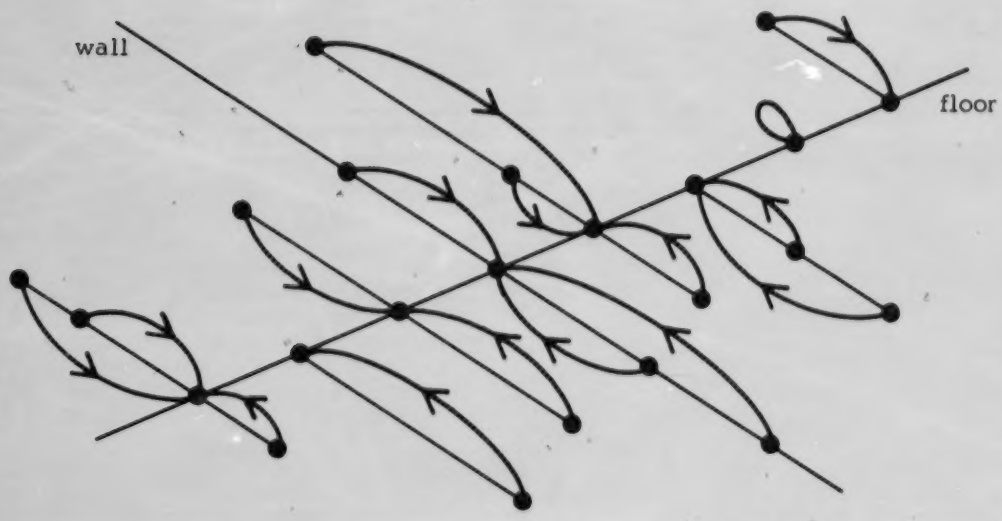
2G

Emphasize that the projection is <u>onto</u> the floor and <u>parallel</u> to the wall. Use a large number of points for this exercise; in particular, several times use a set of points all of which lie on a single line, parallel to the line of the wall, as shown in the illustration. The points on the line of the wall require no special treatment. In fact, the line of the wall determines the projection of any point on it to be its intersection with the floor line.

<u>Exercise 3</u>

T: Now suppose that both the wall and the floor are slanted.

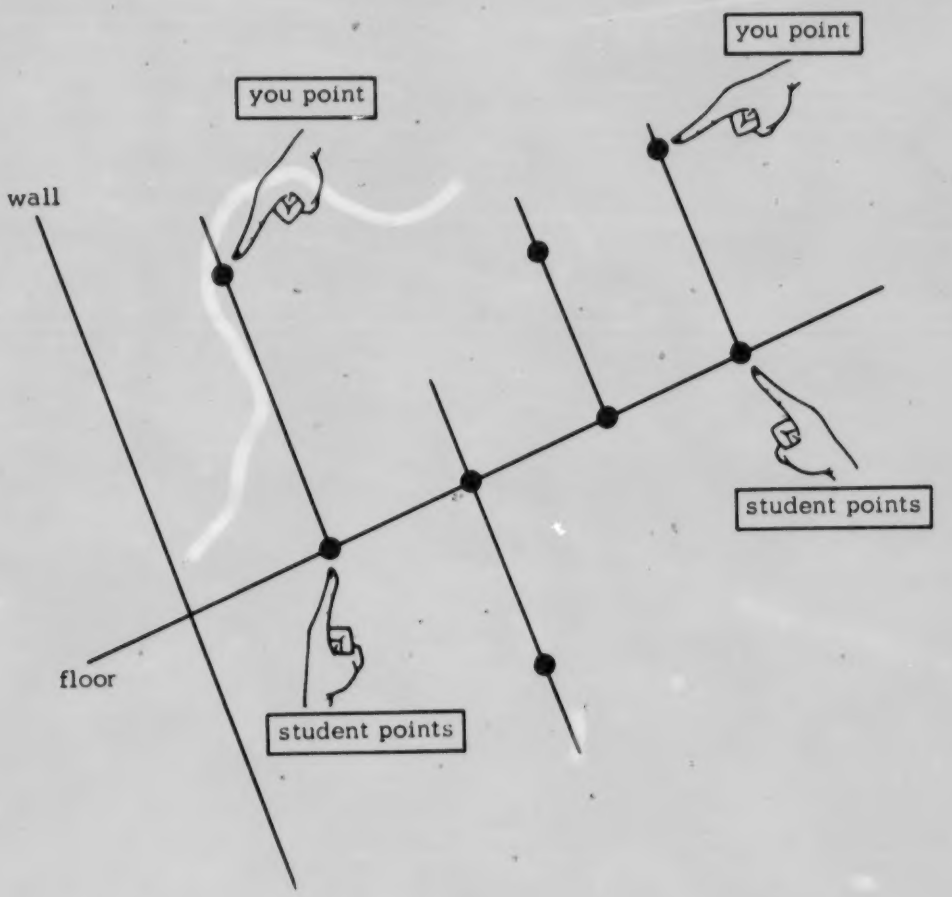
Repeat the exercise with a drawing similar to the following. Sometimes locate a dot and ask students to point to its image; other times point to a dot on the floor and ask students to locate points whose image is this one.



Some discussion may arise now that the orientation of the wall of the drawing does not correspond to that of the wall in the classroom. Emphasize that the projection is parallel to the wall of the drawing.

Exercise 4

Now play "Follow My Finger". In this game, you point to one of the dots in the drawing. A student plays opposite you and points to the image of the dot to which you are pointing. As you move your finger from dot to dot, the student follows by pointing to the corresponding images. When the game is well understood, start to mix in points whose images are not explicitly shown in the drawing. This technique provides you with a check on the students' grasp of the idea of parallel projection. If there is any dispute over the location of the image of a point, perform the construction with a translator.

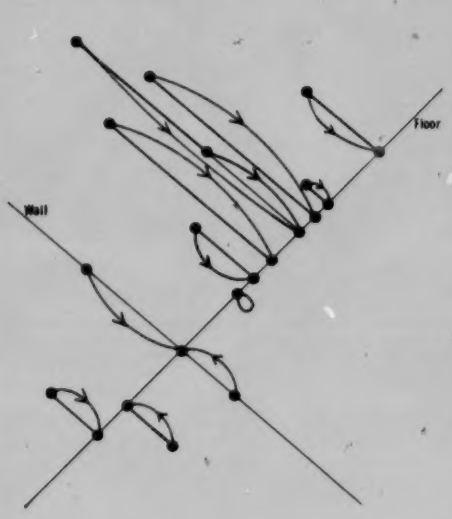


2G

Provide translators and distribute worksheets G13 and G14 for individual work. Monitor the students' work to be certain of proper usage of the translator in finding the image points. Also check that projections are being made onto the floor and parallel to the wall and not the reverse. Each problem is printed on both sides of the worksheets to provide a second chance to any who go astray in their constructions. If there is general difficulty with the problem of G13, stop for a few moments of collective work. Using the last drawing on the board, indicate points on the floor line. Each time, ask a student to find a point whose image is the one indicated by you. Demonstrate the construction with the translator and then return to the individual work.

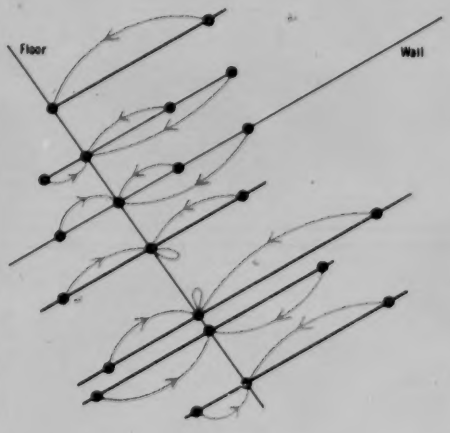
Name _____ G13 *

Project each point onto the floor parallel to the wall.
Draw a red arrow from each point to its image.



Name _____ G14 **

Each point on the floor is the image of many points when they are projected onto the floor parallel to the wall. An example of three points with the same image point on the floor is shown for you.
For each point on the floor, find many points all having this point as image.
Draw a blue arrow from a point to its image.



2.2 Projection #2

NOTE: For this lesson, use the same rubber ball which you used for Lesson 2.1.

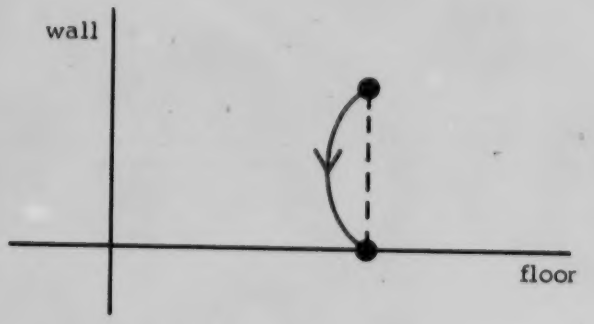
Present your rubber ball again and begin bouncing it on the floor, each time catching it after one bounce.

T: What is the path of the ball as it leaves my hand until it hits the floor?

S: When you drop the ball, it falls straight to the floor.

S: Parallel to the wall.

Illustrate the students' responses.



T: The red arrow is from the point to its image when it is projected onto the floor parallel to the wall. The red arrows are for <u>parallel projection</u>.

Erase the board and draw the wall and floor lines on a slant similar to those in the next illustration (see figure A). Draw dots for many points in various locations (see figure A) and solicit students to indicate the image of each point on the floor after parallel projection. Then use your translator to find each image point; draw red arrows for the parallel projection (see figure B).

Figure A

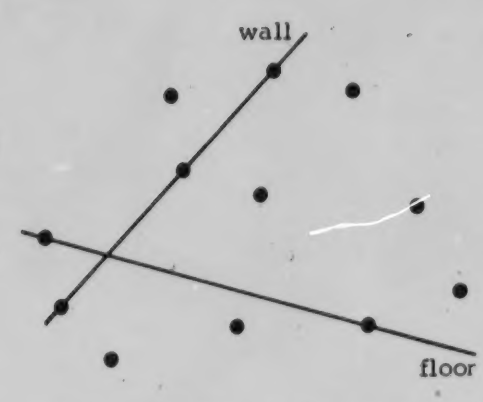
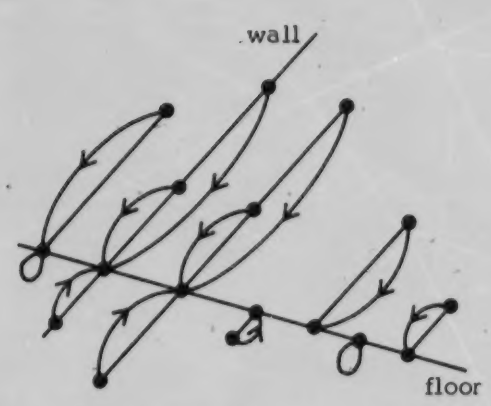


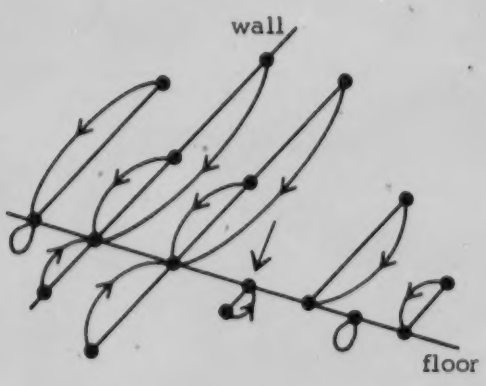
Figure B

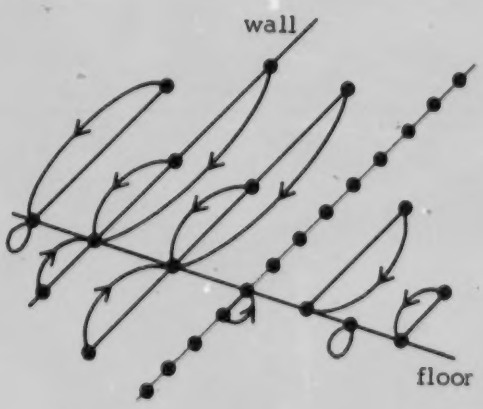


Choose one of the image points on the floor line and point to it.

T: Who can find other points whose image is this point?

NOTE: If you pick the point at the black arrow, then any point on the line containing the blue segment is a correct answer.

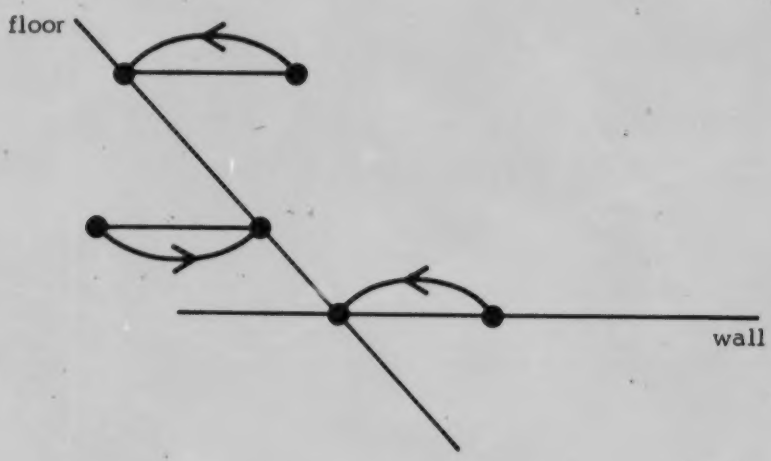




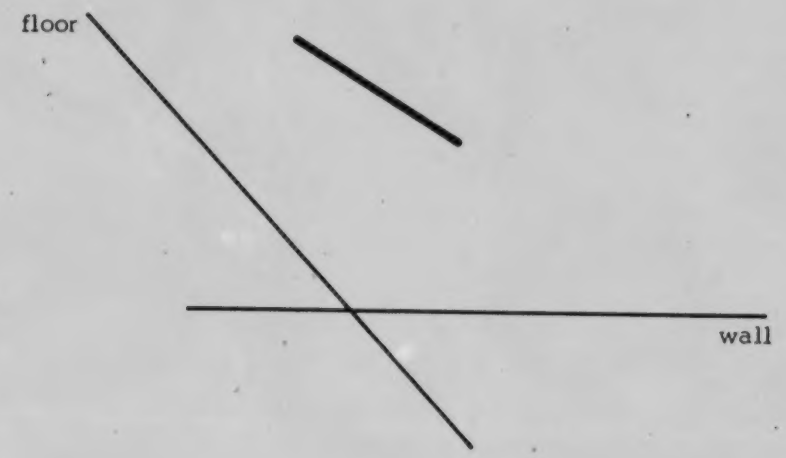
Direct the discussion to a complete solution of the problem. Use the translator to check the parallel projection of each suggested point and continue to ask for more solutions until you have received some on both sides of the floor and the one on the floor itself. Ask how many other points project to this point on the floor to raise informally the question of an infinite number of points on a line, but do not press for any other than general discussion.

4 G

Erase the board and draw another wall and floor as in the next illustration. Locate two or three points and ask students to find their images by projecting onto the floor parallel to the wall. You may have to point out which is the floor line and which is the wall line. Use the translator to check the location of each image point.

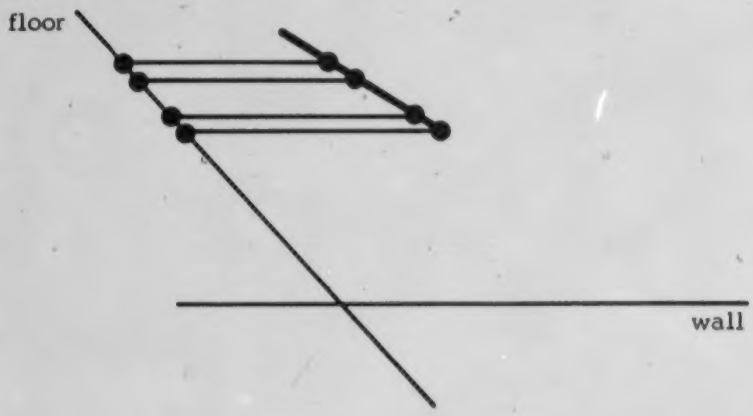


Erase all except the floor and wall. Draw a line segment that does not intersect either the line of the floor or of the wall.

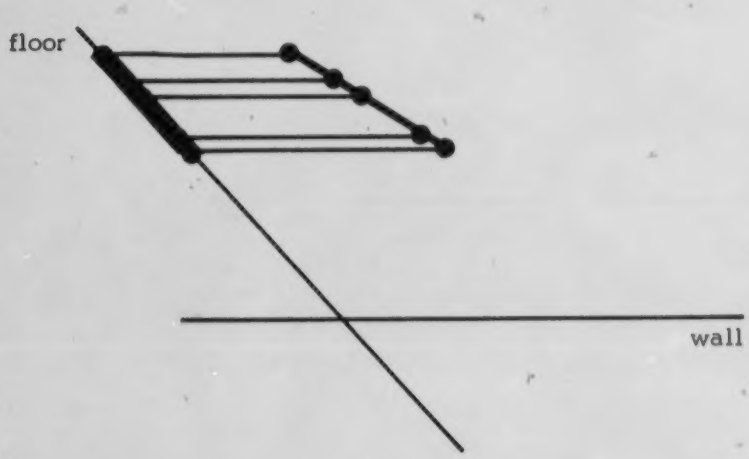


T: There are many points on this line segment. Let's play Follow My Finger to find the projection of some of them onto the floor parallel to the wall.

Play Follow My Finger with the line segment. Occasionally construct the image of one of the points accurately with a translator.

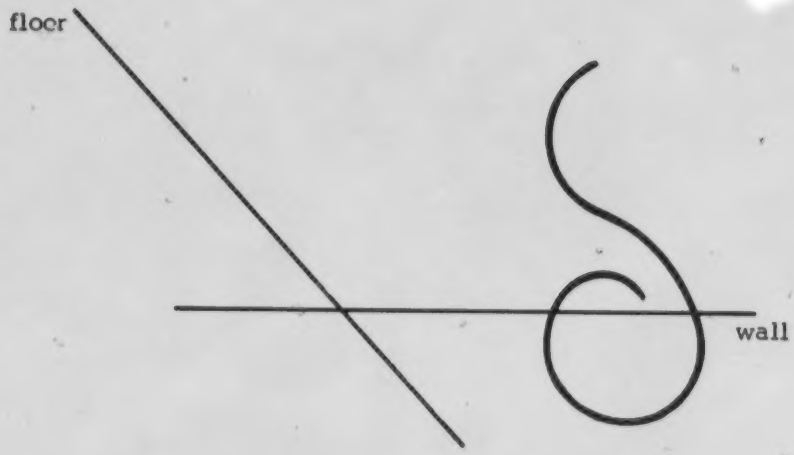


At last, play Follow My Finger by moving your finger continuously over the line segment instead of jumping from point to point. The student must move continuously to follow you. Pause at the points whose images are constructed so that the student can check orientation on the floor line. After playing the game, mark all of the image points with a band of colored chalk.

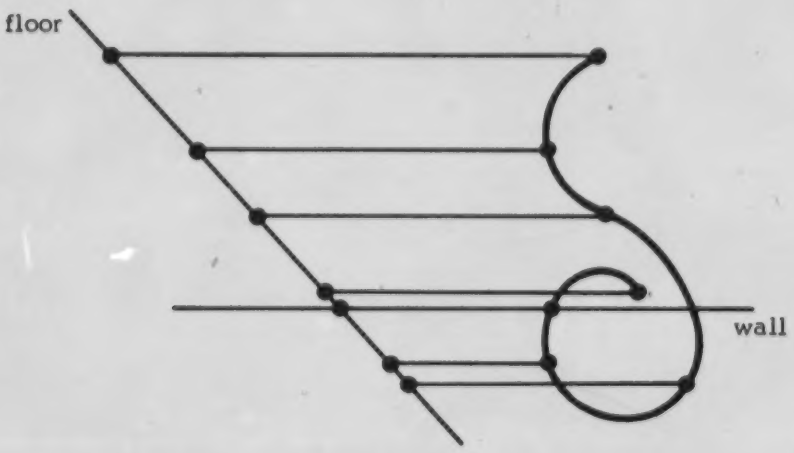


4G

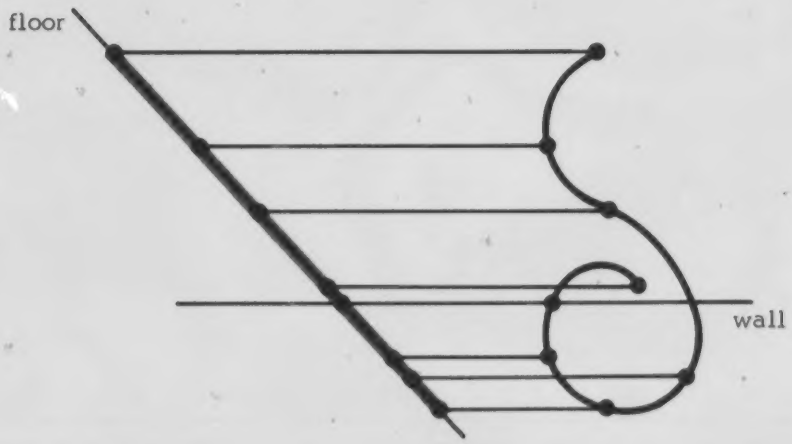
Erase all except the floor and wall. Draw a curve like the one in the next illustration.



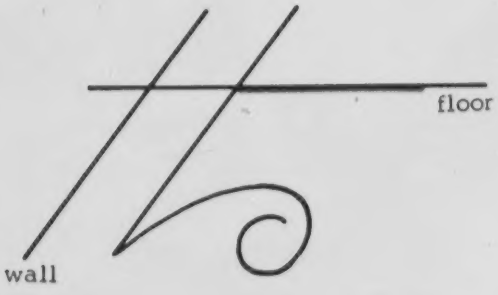
Ask students to suggest points on the curve to project. Ask other students to indicate the images of those points on the floor. Determine the location of the image points accurately with a translator.

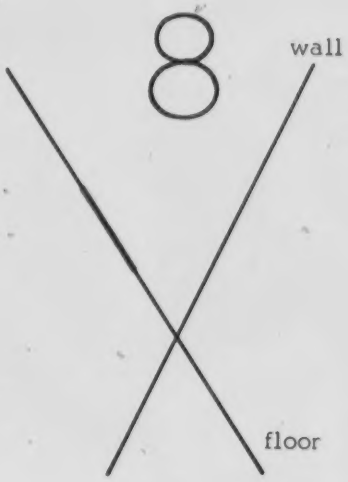


Now play Follow My Finger with several students in turn. The curve provides a challenge to the students in that it requires the follower to reverse directions twice over its length. Begin by jumping among the points whose images are drawn. Then mix in some other points on the spiral. Then take each player in a continuous motion. Concentrate on the spiral at the lower end. Occasionally pause at one of the points whose image is drawn on the board to allow the student to orient properly. Eventually many students will arrive at an understanding of the situation. Then indicate the image of the curve with a colored band.



If interest in the game remains, play Follow My Finger with a few more curves as in the following illustrations or of your own or a students' creation. The image for each curve is shown by the red segment.



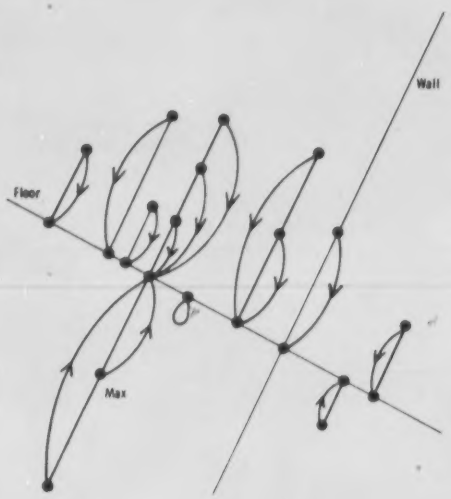


4G

For the remainder of the lesson, let students work independently on Worksheets G15, G16, G17, and G18. Each problem is printed on both sides to give a second chance to those who go astray in their constructions. Encourage any students who finish all four worksheets to create similar problems for each other.

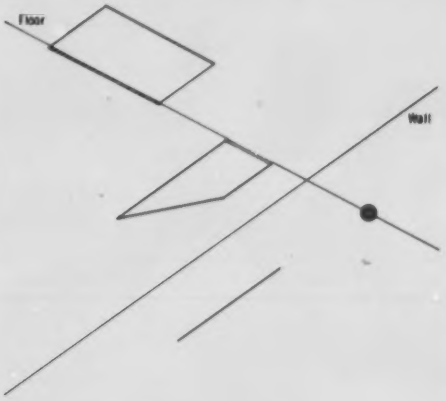
Name _____ **G15** *

Project each point onto the floor parallel to the wall.
Then find some other points that have the same image as Max.



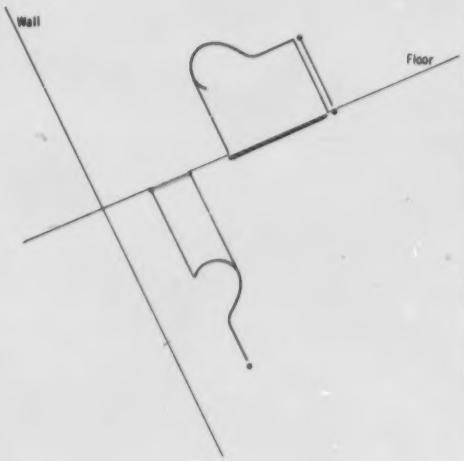
Name _____ **G16** **

Project the three line segments onto the floor parallel to the wall.
Use a different color for the image of each line segment.



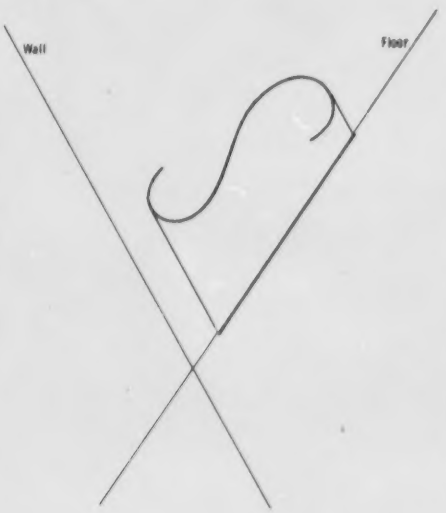
Name _____ **G17** ***

Project each question mark onto the floor parallel to the wall.
Use different colors for the two images.
Be careful, each question mark has two pieces.



Name _____ **G18** ****

Project this graceful curve onto the floor parallel to the wall.
Show the image in red.



5 G

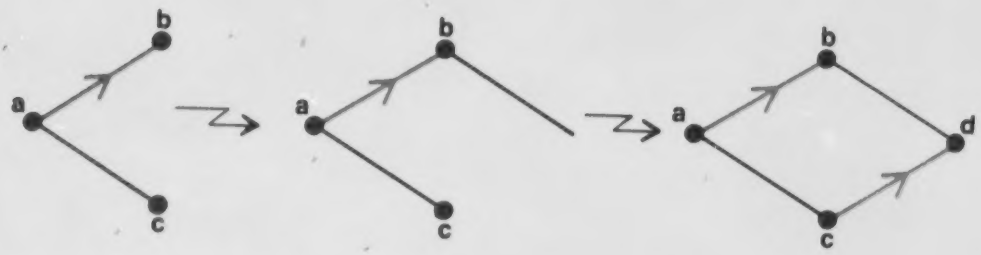
2.3 Couples and Equipollence

Before teaching this lesson, review the discussion of the translator and the basic parallelogram construction in the introduction to this chapter. This lesson includes the first use of the parallelogram construction in this semester, so you may need to amplify some of the discussion depending on how easily your class recalls the details of the construction. A similar comment applies to the use of couples and equipollence.

Draw and label a couple and a third point as shown in this illustration.

T: I want to find another point to make a red couple <u>starting</u> at c that is equipollent to (a,b).

Invite a student to trace the relevant parallelogram. Then carry out the construction with your translator while asking the class for instructions on using the tool. Since this is the first occurrence of the parallelogram construction in this semester, you may have to initiate most of the construction steps yourself.



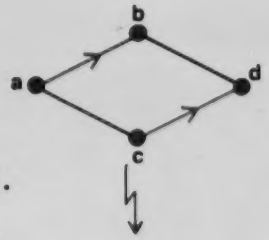
T: Any point can be the starting point of a red couple equipollent to (a,b).

Continue to add points to the picture on the board and to construct couples equipollent to (a,b). The discussion might go on as in the following dialogue. The next illustration indicates progressively how the picture on the board will look.

T: Let's add another point here and call it "x". I want to construct another red couple <u>starting</u> at x. Where should I place the translator? What is the rule?

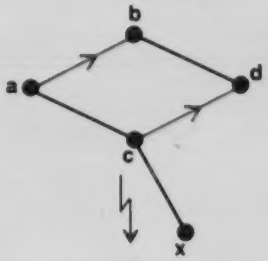
S: Starting point to starting point.

S: Draw a line segment from c to x.
Align the translator with this line and roll.



T: How far should I roll?

S: Roll to d.
Now draw another segment starting at d.

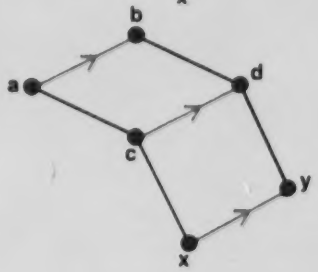


T: What color should I use?

S: White (or the color that you are using in addition to red).

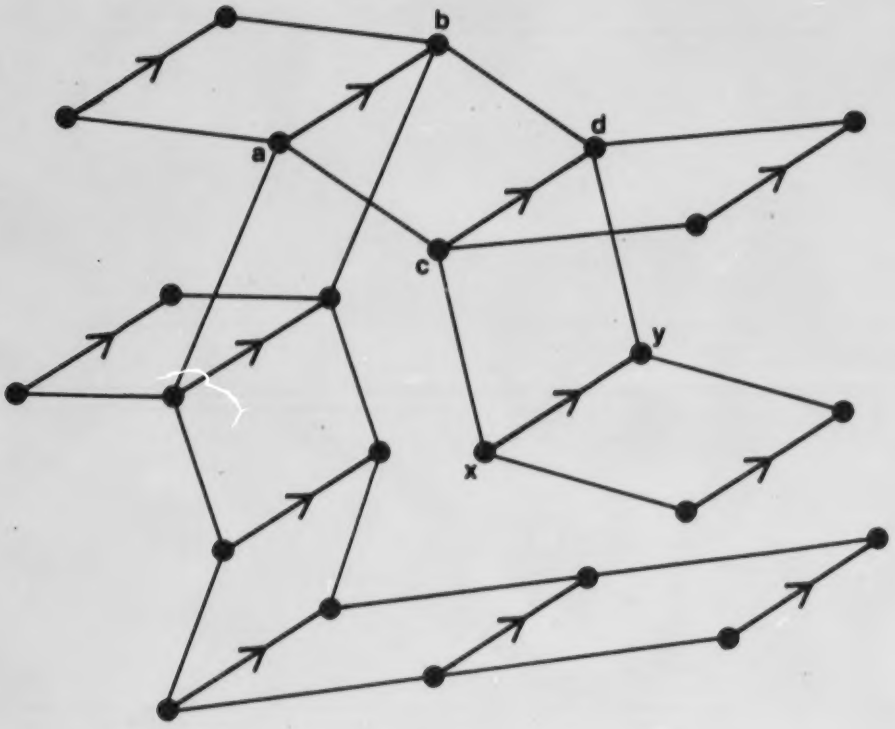
S: Now align the translator with the arrow from c to d, and roll to x.

S: Draw a line segment starting at x.
Draw a <u>red</u> arrow.
x is the starting point of the arrow.



5 G

Constantly emphasize the rules of the construction, starting point to starting point and ending point to ending point. Eventually, after you have put many additional starting points on the board, your drawing should look something like the following illustration.



Now call on several students to play Follow My Finger with you. Use the collection of red couples and each time you point to a starting point, ask the student to point to the ending point.

After several students have played the game with these easy to follow points, change your approach. Indicate a point that was not used as a starting point, then ask the student to follow your finger. The object is to ascertain that yet another red couple must be drawn to locate accurately the student's point.

T: Any point can be the starting point of exactly one red couple. There are no guilty points. The red arrows determine a function. This special function is called a translation. (Write the word on the board.) The image of the starting point of a couple (point to one) is the ending point of the same couple (point to it).

Play Follow My Finger again briefly using the picture of the red couples. Each time ask "What is the image of this point?"

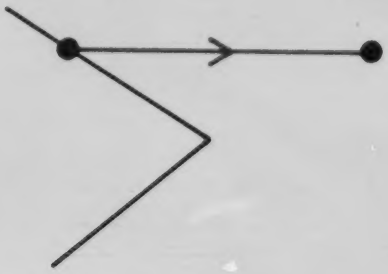
Distribute large pieces of paper and translators for individual drawings. Instruct the students to draw two dots about five centimeters apart, to connect them with a red arrow to make a couple, and then to find six more couples equipollent to that couple. That is, to make a drawing similar to the one just constructed on the board. Monitor the work closely to assist in constructing the parallelograms when necessary.

7G

2.4 Translation #1

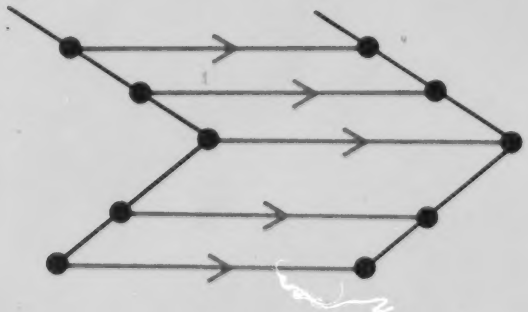
Draw a large two piece zigzag on the board.

T: There are many points on this zigzag. I will pick one at which to start a blue couple.



T: Let's draw more couples equipollent to the blue couple, but with all of their starting points on the zigzag.

Ask students to pick points on the zigzag at which to start couples, and then to trace the blue arrows to the ending points of the appropriate couples. As a check that each of the suggested couples is equipollent to the first blue couple, ask students to trace a parallelogram to show the equipollence. Then perform the construction with your translator. Your drawing may evolve as follows.



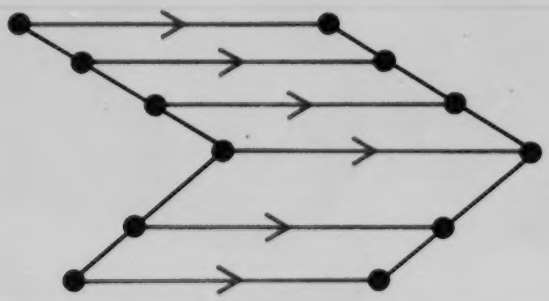
T: The ending points begin to make a picture. What do they look like?

S: They make a zigzag.

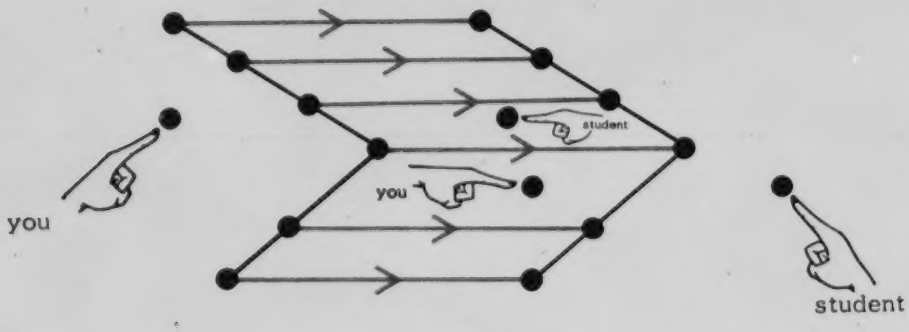
S: The new zigzag is just like the old zigzag.

T: The new zigzag is the image of the original zigzag.

If the entire image of the zigzag is not apparent, complete it with the help of more students.



Play Follow My Finger with two or three students. Pick a point on the original zigzag and ask a student to indicate the image point on the new zigzag. Then use some other points not on the zigzag.



After a while, continue the discussion.

7G

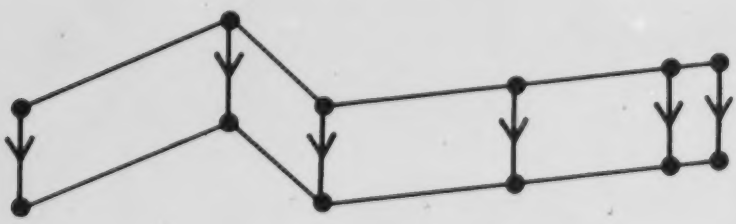
T: Remember that every point on this zigzag (trace the original zigzag) can be the starting point of exactly one blue couple. The blue couples determine a translation function; there are no guilty points. The image of the starting point of a couple (point to one) is the ending point of the same couple (point to it).

Play Follow My Finger again briefly. Each time ask, "What is the image of this point?"

Repeat the exercise with another zigzag and couple.

T: Let's translate this zigzag according to the translation function determined by the red couple.

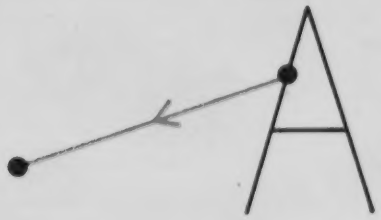
As you perform the construction, ask for the image of each point and mention that you are finding the image of the zigzag. Use Follow My Finger as motivation for constructing more red couples with your translator.



Of course, it is not necessary to find the image of every point of the zigzag by doing the parallelogram construction. Through experience it will occur to the students that there is an efficient choice of points for which to find the images. The most efficient choice of points is the endpoints together with the corners of the zigzags. Some students will realize this quickly, others will not. However, there is no need to press for efficiency.

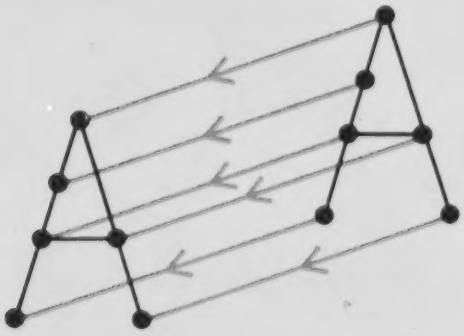
For a few minutes, play the reverse of Follow My Finger; that is, you pick a point on the image zigzag and the student picks the corresponding point on the original zigzag.

Next, draw a letter "A" and a couple as follows.



T: The blue couple determines a translation. Let's find the image of the "A".

Continue as with the zigzags until the image is determined. These drawings can become crowded with lines, so attempt to limit the number of image points needed to see that the image of the "A" is determined. Ask students to trace parallelograms and construction lines before you draw them.



7G

Play Follow My Finger, both versions, for a little while, using the "A". Distribute translators and (progressively) Worksheets G19, G20, G21, G22, and G23 for individual work. The problem of each worksheet is printed on both sides of the sheet to provide a second chance for those who go astray in a construction.

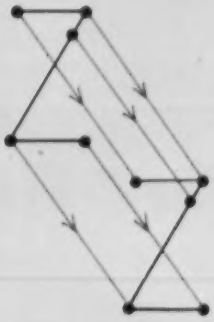
Name _____ G19 *

Find the image of " L " by the translation determined by the red couple .

(Many choices of couples could be used in finding the image. Only one is shown here.)

Name _____ G20 **

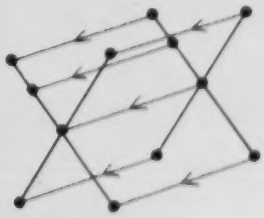
Find the image of " Z " by the translation determined by the blue couple .



(Many choices of couples could be used in finding the image. Only one is shown here.)

Name _____ G21 ***

Find the image of " X " by the translation determined by the red couple.



(Many choices of couples could be used in finding the image. Only one is shown here.)

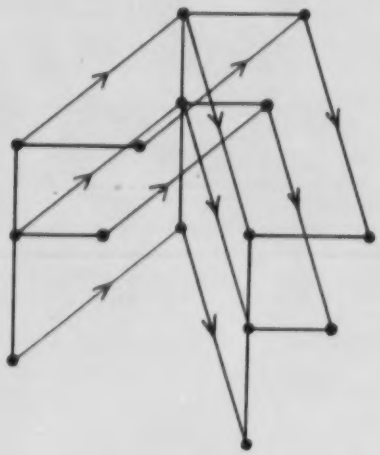
Name _____ G22 ****

Find the image of " K " by the translation determined by the blue couple.

(Many choices of couples could be used in finding the image. Only one is shown here.)

Name _____ G 23 *****

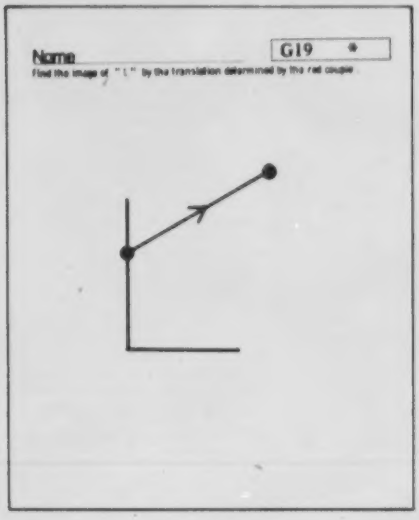
First find the image of "F" by the translation determined by the blue couple. Then find the image of the image of the original "F" by the translation determined by the red couple.

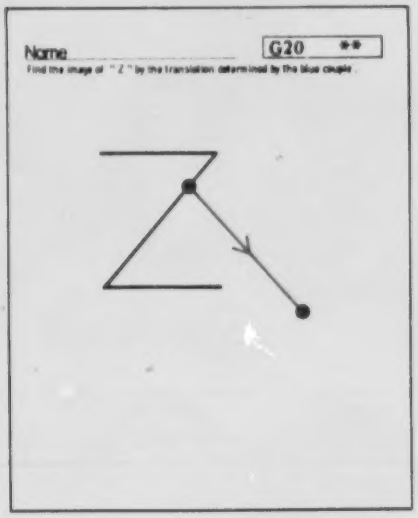


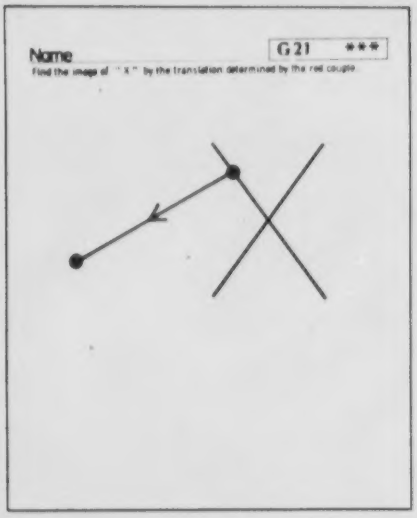
(Many choices of couples could be used in finding the images. Only one is shown here.)

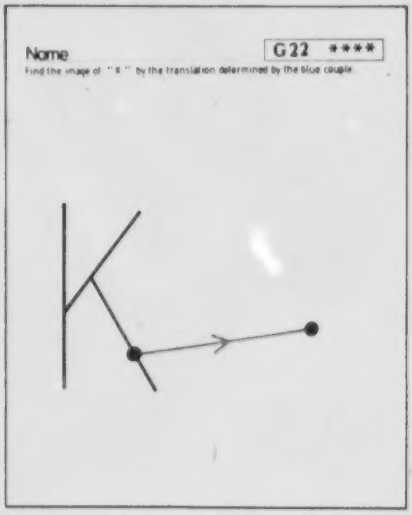
2.5 Translation #2

Lead a collective solution of the constructions of the worksheets from the previous lesson, Worksheets G19, G20, G21, and G22 reproduced below. In each case, copy the drawing onto the board and do the construction at the board with your demonstration translator.







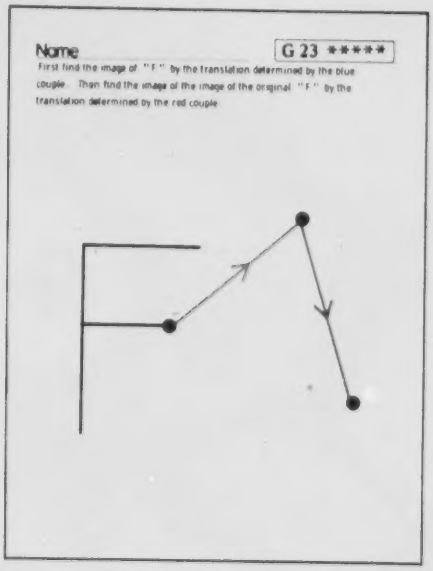


8G

Many students will not have completed all of the worksheets. During the collective discussion of the constructions, lay emphasis on the critical points, corners and endpoints, by making special notice that they all are eventually chosen as starting points for couples. In each case, make a heavy tracing on the image to make it stand out from the construction lines. One solution for each is given in the answer key on page 77.

Even with a collective discussion of the problems, students will benefit from performing the constructions individually on their worksheets. Allow twenty or thirty minutes for them to work on any of these four worksheets which they have not finished earlier. Encourage any who finish to create similar problems for each other.

Next, lead a discussion and collective solution of Worksheet G23. This problem is more complex only because of the greater number of construction lines necessary. One possible completed drawing is shown in the answer key on page 78.

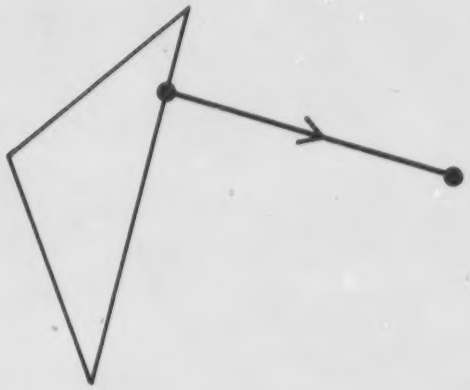

Name
G 23 *****
First find the image of "F" by the translation determined by the blue couple. Then find the image of the image of the original "F" by the translation determined by the red couple.

Ask students to complete this worksheet if they have not done so earlier. Encourage students who finish to create problems for each other similar to the one on this worksheet.

9G

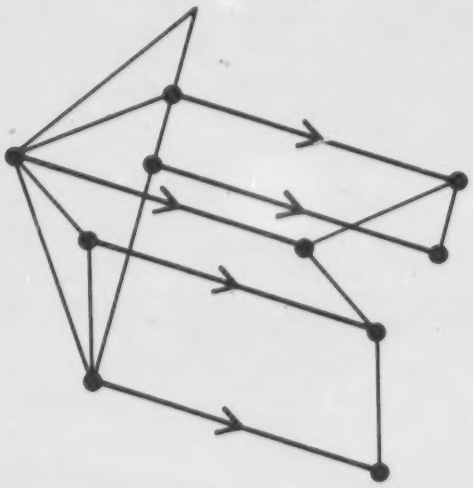
2.6 Translation #3

Draw a triangle on the board and a red couple as shown here.

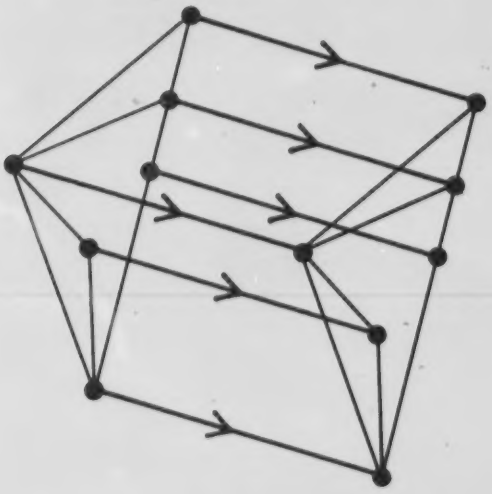


T: The red couple defines a translation. Let's see if we can find the images of all of the points of this triangle.

Invite students to play Follow My Finger. Sometimes ask a student to trace the parallelogram required to construct accurately an image point, that is, to construct a red couple starting on the triangle. After you have found by construction several image points, your drawing may look like this.



Lead the class to the completion of the exercise by emphasizing the choice of critical points, that is, corner points. Some students will recognize them spontanteously; others will not. Some students may continue to choose points on the sides of the triangle or in its interior. There is no need to press for efficiency in these exercises; efficiency will develop through experience.



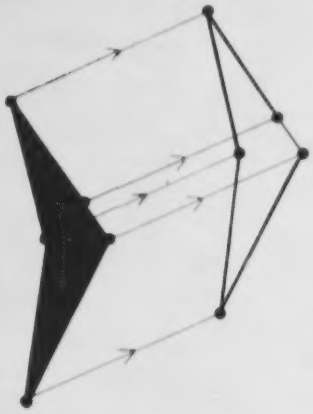
When you have completed the construction, make a heavy tracing of the image of the triangle so that it stands out from the construction lines.

Play Follow My Finger, both versions, for awhile to check the class' understanding of image points.

Distribute Worksheets G24, G25, and G26. Note that each exercise is printed on both sides of the page to provide a second start if a student should go astray in the construction. Encourage those who finish early to create problems of the same type for each other.

Name _____ G24 *

Find the image of the boomerang by the translation determined by the red couples.



(Many choices of couples could be used in finding the image. Only one is shown here.)

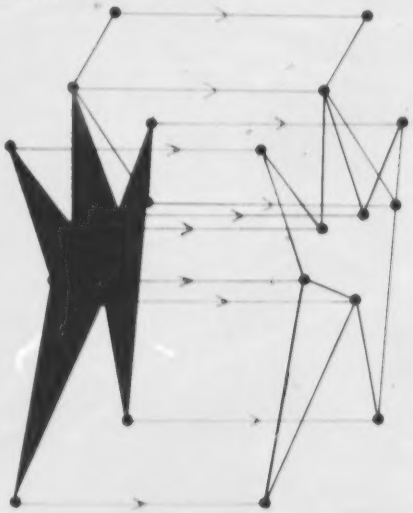
Name _____ G25 **

Find the image of "V" by the translation determined by the red couples.

(Many choices of couples could be used in finding the image. Only one is shown here.)

Name _____ G26 ***

Find the image of the star by the translation determined by the red couples.

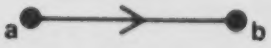


(Many choices of couples could be used in finding the image. Only one is shown here.)

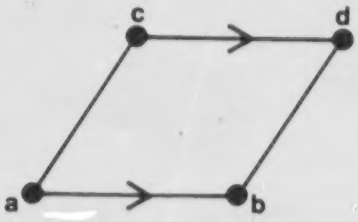
2.7 Lattice Construction

Lessons 2.5 and 2.6 in the Geometry Stand of CSMP Mathematics for the Intermediate Grades, Part II, introduced a construction of a number line using a translator. In this lesson, we review that construction and extend it to the construction of a lattice. The construction anticipates and is a compelling support for future lessons in the Geometry Strand of CSMP Mathematics for the Intermediate Grades, Part IV.

Draw and label a couple and a third point on the board.

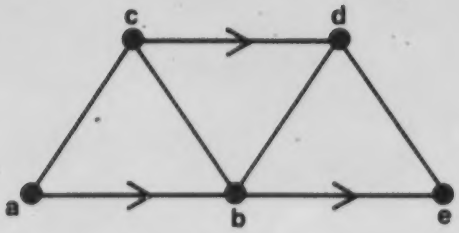


Solicit help from students in constructing another red couple starting at c equipollent to (a,b). As usual, request a student to trace the parallelogram before you do the construction, and ask the class to participate orally in the construction.

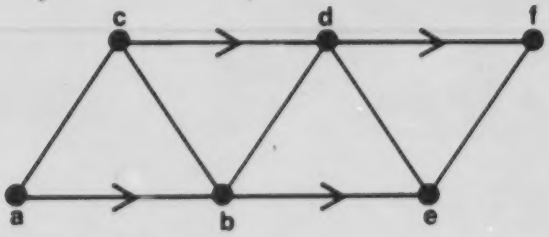


Continue by constructing another red couple starting at b,

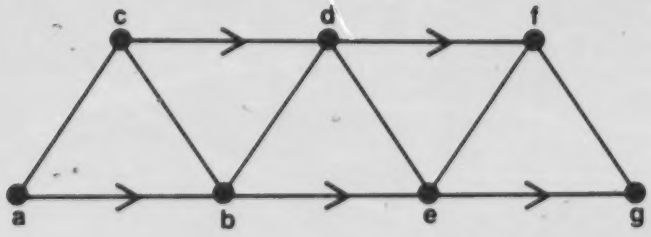
13 G



and then one starting at d,

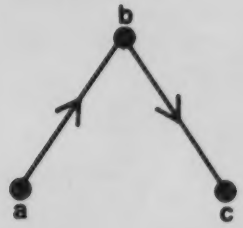


and lastly one starting at e.



Throughout this exercise emphasize the rule guiding the construction of parallelograms: starting point (of a couple) to starting point and ending point to ending point.

Erase the board and draw a pair of couples, one blue and one red, as follows.



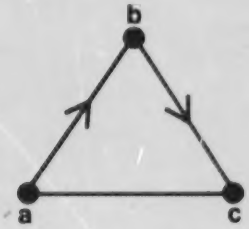
T: Let's draw a red couple <u>starting</u> at c. Which is the starting point of the red couple (a,b)?

S: a is the starting point.

T: And what is the rule?

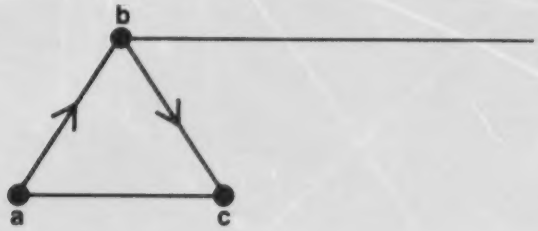
S: Starting point to starting point.

T: Then we connect a to c.



T: What should we do next?

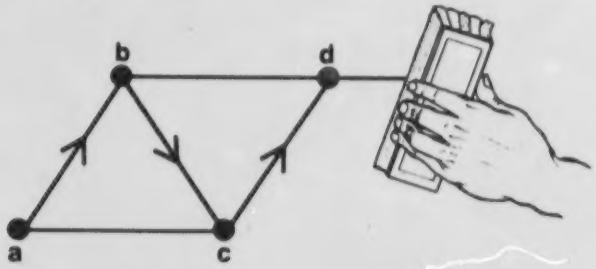
S: Align the translator along the line segment between a and c and roll the translator up to b.



13G

T: Next?

S: Align the translator with the red arrow and roll to c.



T: Now let's draw a blue couple starting at d.
Which is the starting point of the blue couple (b,c)?

S: b.

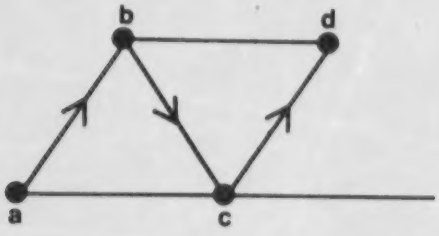
T: And what is the rule?

S: Starting point to starting point.

S: We already have a line segment connecting the two starting points, b and d.

T: What should we do next?

S: Align the translator along the line segment from b to d and roll to c.

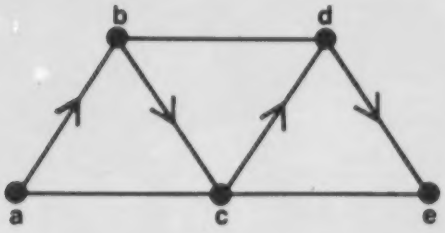


T: Next?

S: Now align the translator with the blue arrow and roll to d.

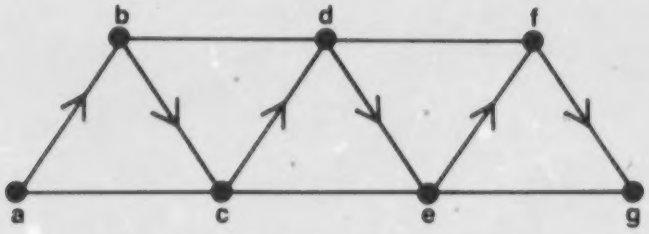
T: What color should we use?

S: Blue rolls to blue.



Continue this construction to add:

1. a red couple (e,f) starting at e, and
2. a blue couple (f,g) starting at f.



T: Now let's draw a red couple <u>ending</u> at c. Which other red couple should we use?

S: We can use the red couple (a,b).

T: Which is its ending point?

13 G

S: b.

T: What is the rule?

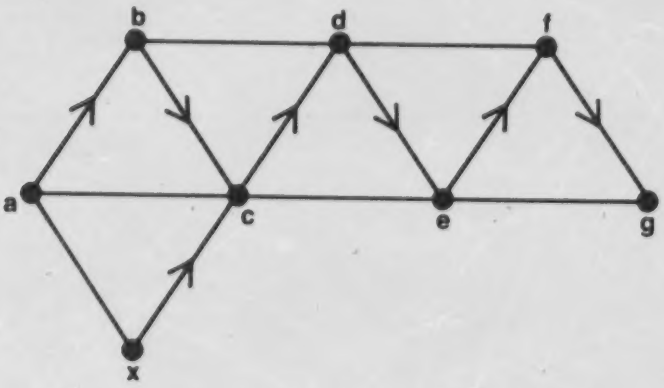
S: Ending point to ending point. We already have a segment connecting b to c, so align the translator along the segment from b to c and roll the translator to a.

T: What color should we use?

S: Blue, we roll from blue to blue and from red to red.

T: Next?

S: Align the translator with the red couple (a,b) and roll to c. Red rolls to red.



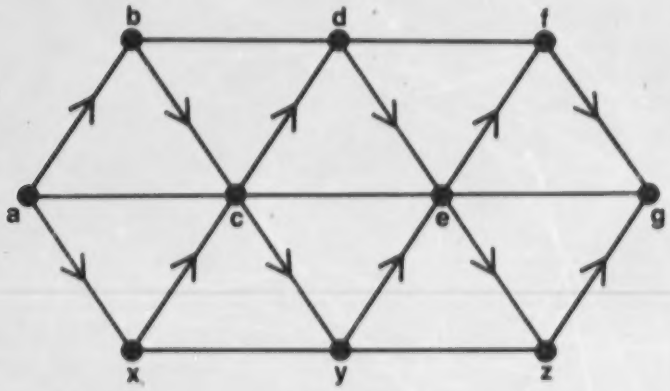
S: The blue segment needs an arrow head.

S: Yes, (a,x) is equipollent to (b,c).

Add the arrowhead.

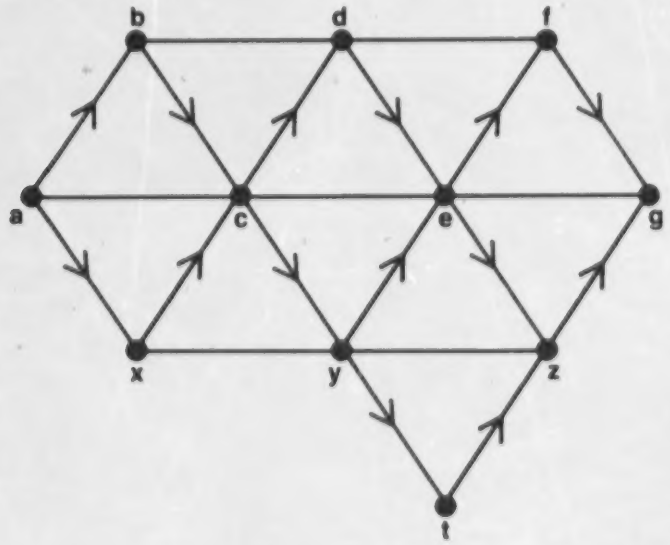
Continue this construction to add:

1. a blue couple (c,y) starting at c,
2. the red couple (y,e),
3. a blue couple (e,z), starting at e,
4. the red couple (z,g).



T: Now, let's draw a red couple ending at z.

Solicit student help to draw a blue couple (y,t) starting at y and then the red couple (t,z) ending at z.



13 G

T: Name the red couples in this drawing.

S: (a,b), (c,d), (e,f), (x,c), (y,e), (z,g), (t,z).

T: Name the blue couples in this drawing.

S: (b,c), (d,e), (f,g), (a,x), (c,y), (e,z), (y,t).

T: Name some other couples.

Affirm all correct answers; any pair of points determines a couple.

T: Who can find some equipollent couples?

S: All of the red couples are equipollent to each other.

S: All of the blue couples are equipollent to each other.

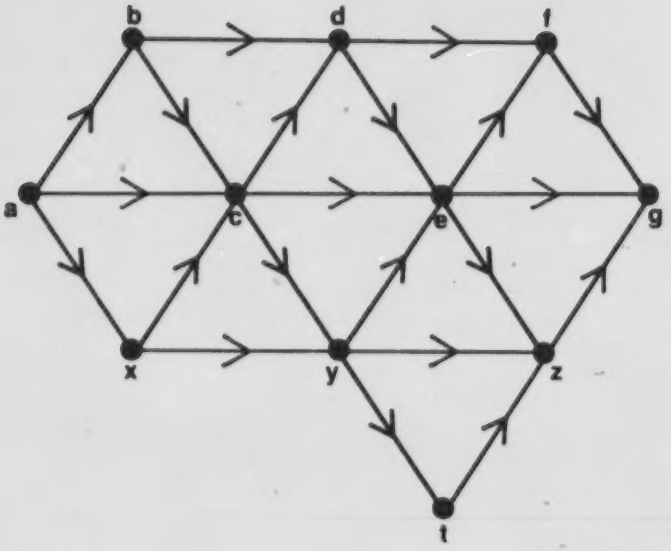
T: Are there any others?

S: (a,c) is equipollent to (b,d).

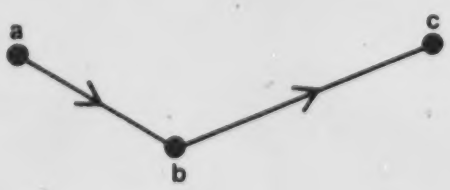
T: Are there any other couples equipollent to them?

S: (d,f), (c,e), (e,g), (x,y), (y,z).

Add arrowheads to the line segments connecting these couples as they are mentioned.



Distribute large sheets of unlined paper and individual translators. On the board, draw another pair of couples as illustrated below. Instruct the students to draw red and blue couples on their papers as you have drawn on the board, and then to build a drawing similar to the one just drawn.



Tell them to begin by drawing a blue couple starting at c.

Monitor the students' work to help them avoid going astray in their constructions.

PROBABILITY AND STATISTICS

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part III

Probability and Statistics

CEMREL, Inc.

TABLE OF CONTENTS

SUGGESTED SCHEDULE FOR P-LESSONS iv

SUGGESTED CHRONOLOGICAL ORDER OF P-LESSONS . . . vii

INTRODUCTION ix

CHAPTER 1: SHUNDA'S NEWSSTAND 1

1.1 Shunda's Newsstand #1 2

1.2 Shunda's Newsstand #2 7

1.3 Shunda's Newsstand #3 12

1.4 Shunda's Newsstand #4 18

SUGGESTED SCHEDULE FOR P - LESSONS

WEEK	Probability and Statistics	Materials
1	1L L1.1 Composition of Relations p. 5	Teacher: Colored chalk; IG-III Language Poster #1 Student: Colored pencils; Worksheets L01, L02, L03
2	2L L2.1 String Game Analysis #1 p. 90	T: Felt-tipped pen; The String Game; 2 Demonstration Analysis Sheets S: Pencil; String Game Analysis Pad; Worksheets L18, L19, L20, L21
3	3L L1.2 Addition and Multiplication with Ten Friends p. 21	T: None S: Worksheets L04, L05
4	4L L1.3 Powers of ●2 p. 32	T: Colored chalk S: Colored pencils; Worksheets L06, L07
5	5L L2.2 String Game Analysis #2 p. 95	T: Colored chalk; The String Game S: String Game Analysis Pad
6	6L L1.4 Arrow Messages p. 43	T: Colored chalk S: Colored pencils; Worksheets L08, L09, L10, L11
7	7L L1.5 The Kite p. 52	T: Colored chalk S: Colored pencils; Worksheets L12, L13
8	ADJUSTMENT DAY	
9	8L L2.3 String Game Analysis #3 p. 103	T: Colored chalk; The String Game S: String Game Analysis Pad; Worksheets L22, L23, L24, L25

SUGGESTED SCHEDULE FOR P - LESSONS

WEEK	Probability and Statistics	Materials
10	**9L** **L1.6 Addition and Multiplication with Eighty Friends** **p. 60**	**T: None** **S: Worksheets L14, L15**
11	ADJUSTMENT DAY	
12	**10L** **L2.4 String Game Analysis #4** **p. 111**	**T: Felt-tipped pen; colored chalk; two Demonstration Analysis Sheets; The String Game** **S: String Game Analysis Pad; Wkshts. L26, L27, L28, L29, L30**
13	**11L** **L1.7 The Kite with Four Tails #1** **p. 72**	**T: Colored chalk** **S: Paper; colored pencils**
14	**12L** **L1.8 The Kite with Four Tails #2** **p. 78**	**T: Colored chalk; IG-III Language Poster #2** **S: Paper; colored pencils; Worksheets L16, L17**
15	**1P** P1.1 Shunda's Newsstand #1 p. 2	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"
16	**2P** P1.2 Shunda's Newsstand #2 p. 7	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"
17	**3P** P1.3 Shunda's Newsstand #3 p. 12	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand"
18	**4P** P1.4 Shunda's Newsstand #4 p. 18	T: Colored chalk; "Shunda's Newsstand" S: Paper; colored pencils; "Shunda's Newsstand

SUGGESTED CHRONOLOGICAL ORDER OF P - LESSONS

Lesson One	P1.1 - Shunda's Newsstand #1.	2
Lesson Two	P1.2 - Shunda's Newsstand #2.	7
Lesson Three	P1.3 - Shunda's Newsstand #3.	12
Lesson Four	P1.4 - Shunda's Newsstand #4.	18

PROBABILITY AND STATISTICS

INTRODUCTION

There are several reasons why people study mathematics. It is an interesting and challenging intellectual exercise when studied for its own sake. But it is also a supremely useful topic of study in that it has applications outside itself--to other disciplines, as well as more generally to understanding the world around us. Some areas of mathematics are more easily applied than others, and the areas which particularly concern us in this strand are towards the easy end of the spectrum, both because of the small amount of background mathematical knowledge required before meaningful application can be made, and because it is usually fairly easy to see the link between the mathematical model and the "real" situation being modeled. For these reasons, it would be difficult to overstate the importance in today's world of probability and statistics and their applications to problems and situations within and outside the discipline of mathematics.

Since situations which involve us in finding the probabilities of certain events give support to our work with fractions, there are lessons on probability in Chapter 7 of the World of Numbers strand (see Sections 7.2 and 7.4). Also some of these situations give rise to combinatorial problems and lessons on such problems can be found in Chapter 3 of the World of Numbers strand (see Sections 3.5 and 3.6).

This semester the lessons of this strand are of a purely statistical nature, involving methods of recording data and making decisions based on the data. These lessons are presented in a very realistic situation and suggest a genuine applied mathematics problem.

1

SHUNDA'S NEWSSTAND

A very common problem in the business world is to determine the optimal quantity of items to buy or produce in order to make the most profit. Many factors affect such decisions and no one can be guaranteed of always making the best prediction. Still, reasonable decisions can be made by doing a careful study of sample data. The lessons of this chapter investigate just such a problem: Shunda must determine how many newspapers to buy (how to supply her newsstand) in order to realize the most profit from sales. Prompted by some applied mathematics and stimulating pictures for recording data, the students are led to a solution of the problem.

CAPSULE LESSON SUMMARIES

Section 1.1: Present the story of SHUNDA'S NEWSSTAND and discuss the general situation. Calculate profit from various sales records.

Section 1.2: Review the story, continue to calculate profit from various sales records, and investigate some new pictures for recording data.

Section 1.3: Review the story situation. Present the problem of having to determine a constant daily supply of newspapers. Make a record of the data for a 20-day experimental period and determine the total profit for different choices of daily supply.

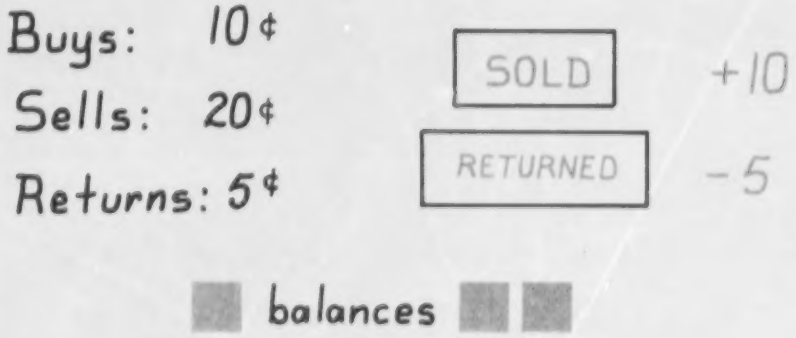
Section 1.4: Review the story and the problem. Investigate the effect on the total profit for the 20-day experimental period of changing the supply. Solve the problem.

1P

1.1 Shunda's Newsstand #1

Begin the lesson by asking the students whether any of them has been a newspaper seller. You should encourage the class to discuss the arrangements that a newspaper seller makes for such a business. Raise such questions as "How much does a newspaper seller pay for papers?"; "How much does a newspaper seller charge customers for papers?"; "What does a newspaper seller do with unsold papers?"; and so on.

Distribute copies of the story-workbook "Shunda's Newsstand". Read pages 2-5 together and write the following information on the board as it occurs in the reading.



Let the students explain how a gain of 10 cents is made when one newspaper is sold, a loss of 5 cents is had when a newspaper is returned, and one newspaper sold balances two newspapers returned. Encourage the students to associate "balances" with "break-even profit" or "gain and loss are the same" or "profit of $0".

Page 6

Let the class first discuss the page. They should observe how the picture illustrates a supply of 22 newspapers and what were the final sales for the day.

T: How many newspapers did Shunda sell this day?

S: 10; there are 10 blue squares.

T: How many newspapers did Shunda return to the dealer?

S: 12; there are 12 red squares.

T: Could you find how many were returned without counting the red squares?

S: 10 sold and 22 - 10 = 12; so 12 returned.

T: Very good. Now, let's check the calculations. How do we get a gain of $1.00?

S: 10 newspapers sold and a gain of 10 cents for each newspaper sold makes a gain of (10 x 10 =) 100 cents. That's $1.00.

T: How do we get a loss of $.60?

S: 12 newspapers returned and a loss of 5 cents for each newspaper returned makes a loss of (12 x 5 =) 60 cents.

T: How do we get a profit of $.40?

S: $1.00 - $.60 = $.40.

NOTE: Throughout this story the word "profit" is used to mean the difference between what Shunda takes in for newspapers (either by selling or returning) and what she paid for the newspapers. On the balance sheet the profit is found simply as "gain - loss". In some cases the profit will be negative; for example, when the gain is $.60 and the loss is $1.00, the profit is ¯$.40. This use of the word "profit" may initially bother some students because often "profit" connotes "positive". However, if you use the word consistently and perhaps note that for Shunda's records the profit could be positive or negative, the students should easily adapt. Do not encourage extensive discussion of various meanings of the word as that would only be distracting.

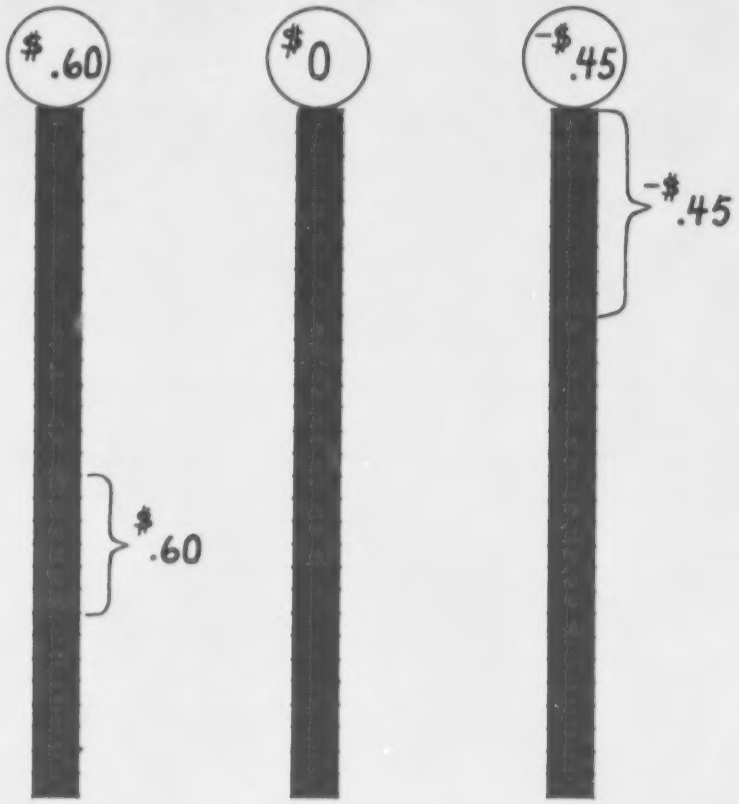
Page 7

Collectively check the calculations that resulted in the three days profit-o-

meters on this page. One way to do this would be to make a balance sheet for each day.

SUPPLY	GAIN	LOSS	PROFIT
30	$1.40	$ .80	$ .60
30	$1.00	$1.00	$ 0
30	$ .70	$1.15	-$ .45

A student might suggest that you could match two red squares with one blue square, and because they balance, cross them off. Then you would only have to consider squares which were not crossed off.



This is a nice method and should be encouraged if it is suggested. If both the balance sheet method and this method are suggested, use one to check the other. However, if this method is not suggested do not force the class to consider it.

Page 8

Let the students independently do the calculations necessary to complete the balance sheet and profit-o-meter on this page. As the class is working, observe and help those students who are having difficulty. When most of the students are finished, check the work collectively. Answers for this page can be found at the end of this chapter on page 21.

Page 9

Collectively fill in the blanks at the top of the page.

Shunda bought 36 newspapers.

She sold 30 newspapers.

She returned 6 newspapers.

Ask the students to independently complete the balance sheet and profit-o-meter. While the students are working, put these problems on the board.

Make a balance sheet and find the profit.

1) Supply : 27
Sold : 20

2) Supply : 27
Sold : 10

3) Supply : 27
Sold : 5

1P

Students who finish the work on page 9 should be asked to solve these three problems. Those students who finish all of these problems quickly could be asked to consider this problem.

4) Shunda has a supply of 27 newspapers. How many must she sell to break-even (that is, have a profit of $0)?

About five or ten minutes before the end of the lesson, collectively solve the problem on page 9 and the problems on the board. Ask the students to check their work. Answers for page 9 are on page 21 at the end of this chapter and solutions for the first three additional problems are given below.

1) Supply: 27
Sold: 20

GAIN	LOSS	PROFIT
$ 2.00	$ .35	$ 1.65

2) Supply: 27
Sold: 10

GAIN	LOSS	PROFIT
$ 1.00	$ .85	$ .15

3) Supply: 27
Sold: 5

GAIN	LOSS	PROFIT
$ .50	$ 1.10	-$ .60

If some students have considered problem 4, ask them to explain their solutions. The solution is 9, because 9 newspapers sold balance 18 newspapers returned. This observation may not be forthcoming; many students will use trial and error until they get a balance sheet with $0 profit.

4) Supply: 27
Sold: 9

GAIN	LOSS	PROFIT
$ .90	$ .90	$ 0

Collect the story-workbooks and have them ready for next week's lesson.

1.2 Shunda's Newsstand #2

Distribute paper, colored pencils, and the student's copies of the story-workbook.

Ask the class to recall the story of SHUNDA's NEWSSTAND that was begun last week. As students mention the important information in the story, write in on the board.

Buys: 10¢
Sells: 20¢
Returns: 5¢

SOLD +10

RETURNED -5

Encourage them to recall that one newspaper sold balances two newspapers returned.

■ balances ■■

Page 9

Recall the calculations that were done last week to complete the balance sheet and profit-o-meter on this page.

Put this problem on the board and ask the students to solve it on their papers.

273

2 P

Make a balance sheet and find the profit.

Supply: 36

Sold: 25

Then check their solutions collectively.

GAIN	LOSS	PROFIT
$2.50	$ .55	$ 1.95

T: Suppose Shunda had a supply of 36 newspapers and she had a negative profit. How many newspapers could she have sold?

S: 0.

S: Just 1 or 2.

T: Could she have sold more and still had a negative profit?

Let the class discuss this question for a minute and encourage them to discover that Shunda could have sold from 0 to 11 newspapers and had a negative profit.

T: What would Shunda's profit be if she sold 12 of the 36 newspapers?

S: $0.

T: Yes, that is the break-even point. If she sold 12, then she returned 24 and

12 SOLD balances 24 RETURNED

Check this result with a balance sheet.

GAIN	LOSS	PROFIT
$1.20	$1.20	$0

Page 10

Briefly discuss the meaning of these zigzag pictures as "stylized" profit-o-meters.

T: Do you know what these pictures are?

S: They are like profit-o-meters but with line segments rather than squares.

T: Why is there "-$.20" in this circle?

S: That is the profit when Shunda sells 3 newspapers (3 blue segments) and returns 10 newspapers (10 red segments).

Check this result in a balance sheet.

GAIN	LOSS	PROFIT
$ .30	$ .50	-$ .20

2P

Ask the students to complete the profit-o-meters on this page by themselves. Allow about ten minutes for this independent work. Students who finish quickly could go on to solve the problem on page 11.

Check the work on page 10 collectively. It is not necessary that every student finish the page before this checking; those who do not finish can participate in the checking of the problems they have done.

Page 11

Complete this profit-o-meter collectively. Some students may have already solved the problem and they can provide necessary information.

T: This is a very long profit-o-meter. How should we find the profit?

S: Count the blue squares and the red squares, and then make a balance sheet.

T: Yes, we could do that. Has anyone already made these counts?

S: There are 136 blue squares and 129 red squares.

It is likely there will have been some counting errors, but eventually the class should be told these correct numbers.

GAIN	LOSS	PROFIT
$13.60	$6.45	$7.15

T: Did anyone solve this problem another way?

S: I crossed off 1 blue square and 2 red squares, then 1 blue and 2 red, then 1 blue and 2 red, then 1 blue and 2 red, and so on. There were 72 blue and 1 red left over. So the profit is

$7.20 - $.05 = $7.15.

Encourage the students to explore different attacks on this problem to make it easier, but do not spend too long. If necessary, give the correct numbers yourself.

Pages 12 and 13

Discuss the pictures on these pages so that the class understands that they are again "stylized" profit-o-meters.

T: Do you understand the faces?

S: A smiling face means the profit is positive, a sad (frowning) face means the profit is negative, and a "so-so" (neutral) face means $0 profit or break-even.

For the rest of the period, let the students work independently on completing these two pages. You may have to give individual help to some students who have not recognized how to determine what kind of face to draw, but most students will catch on without further explanation.

At the beginning of the next lesson, you will check these pages collectively. At the end of the class period, collect the story-workbooks and have them ready for next week's lesson.

3 P

1.3 Shunda's Newsstand #3

Distribute the students' copies of the story-workbook and colored pencils. Briefly review the story of SHUNDA'S NEWSSTAND, encouraging the class to recall the important information. Write this information on the board as it is mentioned.

Buys: 10¢
Sells: 20¢
Returns: 5¢

SOLD +10
RETURNED -5

■ balances ■■

Pages 12 and 13

Collectively check and discuss these two pages. Review the idea that a smiling face indicates a positive profit, a sad (frowning) face indicates a negative profit, and a "so-so" (neutral) face indicates $0 profit or break-even.

T (pointing to the upper left picture): Why is this first face a "so-so" face?

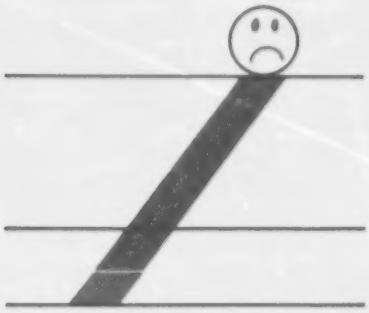
S: Because there was $0 profit; Shunda had a break-even day.

T: How can we be sure from the picture?

S: Because there is twice as much red as blue, and two reds (newspapers returned) balance one blue (newspapers sold).

T: Very good; one-third of the profit-o-meter is blue and two-thirds is red.

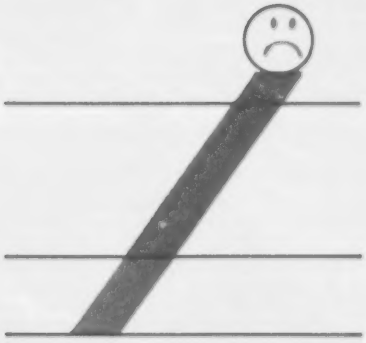
Continue, if necessary, to discuss why certain faces are smiling or frowning. In checking what kind of face should be in a circle, sometimes ask a student to explain. For example,



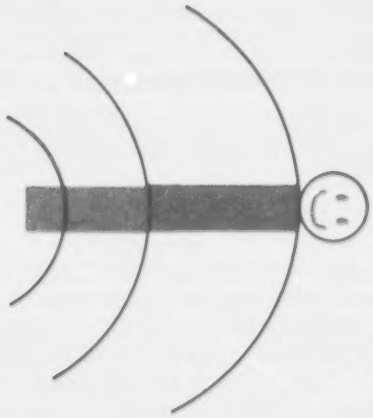
S: It's a sad face because red is below the in-between line.

S: To break-even, blue has to be up to the in-between line, and it is not; so the profit is negative.

S: Red is more than two-thirds and blue is less than one-third.



S: It's a sad face because red is above the top line. To break-even, red would be from the top to the in-between line and blue from the in-between line to the bottom line. But here there is more red (returned), so the profit is negative.



S: It's a smiling face because blue is inside the inner circle. To break-even, red has to be from the outer circle to the in-between circle and blue from the in-between circle to the inner circle. There is more blue (sold) here, so the profit is positive.

3 P

S: It's a smiling face because half of the newspapers were sold and half returned. To break-even or have a negative profit, Shunda would have returned more than she sold. So this must be a positive profit.

NOTE: An example might provide a good explanation in this case.

Supply :	40
Sold :	20
Returned :	20

GAIN	LOSS	PROFIT
$ 2.00	$ 1.00	$ 1.00

Page 14

Read page 14 collectively. Ask the class how Shunda could decide what would be the best constant supply. You might get comments such as:

S: She should buy only as many newspapers as she is sure to sell.

S: She should buy about 17 newspapers because she most often sold 17 newspapers.

S: She should buy about 22 newspapers, because half the time she sold more than 22 newspapers and half the time she sold 22 or less than 22 newspapers.

S: She should buy 20 newspapers because 20 is half way between 5 (the fewest number sold) and 35 (the largest number sold).

S: She should buy the average number sold. (NOTE: In this case that number is 22.)

Page 15

Let the students complete this page independently. They should notice that the bar graph here is another way to record the demand for the 20-day experimental period.

Pages 16 and 17

Collectively read and discuss these pages. The class should observe that these pages are just a reorganization of page 15.

Pages 18 and 19

Read page 18 collectively and discuss how this 20-day balance sheet should be completed. The students might suggest counting all the red squares and all the blue squares on page 19 and then using these numbers to find the total gain and total loss. This method would be correct, but rather time consuming. With some prodding, the students should observe that you can add the numbers of newspapers sold on all 20 days (a hand-calculator would be useful to do this addition rapidly) to get 440. Then to find the number returned, subtract 440 from 700 (20 x 35), the total supply for the 20 days.

SUPPLY: 700

SOLD: 440

RETURNED: 260

3P

With this information, ask the students to complete the balance sheet by themselves and then check their work collectively.

SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
35	$44.00	$13.00	$31.00

Before going on to page 20, ask the class whether they think Shunda could make a better choice than 35 for the constant daily supply. Accept any opinions without comment.

Pages 20 and 21

Read page 20 collectively and as for page 18 discuss how this 20-day balance sheet should be completed. The students might suggest counting the red squares because there are not too many and then subtracting this number from 400 (the total supply) to find the number of blue squares. Accept any reasonable method which leads to this information.

TOTAL SUPPLY: 400

TOTAL SOLD: 351

TOTAL RETURNED: 49

With this information, let the students complete the balance sheet on their own. Then check their work collectively.

SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
20	$35.10	$2.45	$32.65

T: Is a constant daily supply of 20 a better choice than 35?

S: Yes, the profit is greater. Here it is $32.65 and with 35 it is only $31.00.

Let the class again discuss whether there might be a still better choice for the constant daily supply. Again, accept opinions without comment.

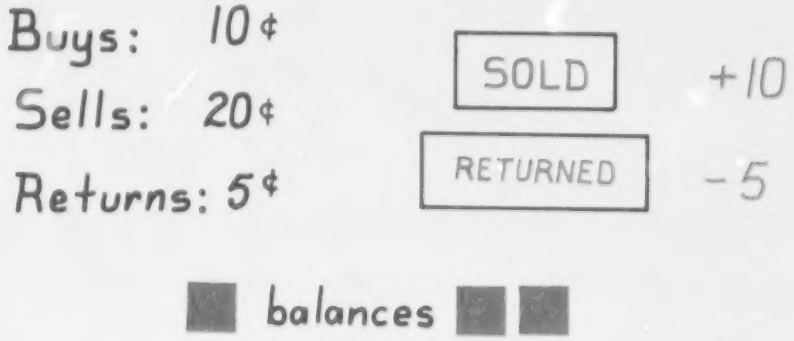
Collect the story-workbooks and have them ready for next week's lesson.

4P

1.4 Shunda's Newsstand #4

Distribute the student's copies of the story-workbook and colored pencils.

Begin by reviewing the important information from the story.



Recall with the class the problem Shunda has of having to choose a constant daily supply.

Pages 18 - 21

Review the record for the 20-day experimental period, how the balance sheets on pages 18 and 20 were completed, and why a daily supply of 20 would be a better choice than 35. Again encourage the students to suggest what they think would be the best choice for a constant daily supply.

Pages 22 and 23

Read page 22 collectively and, with class discussion, check that the profit increase from a supply of 20 to 21 is indeed $.80.

NOTE: Adding this row to the profit-o-meter increases the total gain by $1.20 and increases the total loss by $.40. Therefore, the increase of profit is $.80.

T: Do you see why Shunda sees immediately that changing the daily supply from 21 to 22 would again increase her profit by $.80?

S: The row for 22 is the same as the row for 21.

Ask the students to check this on the graph on page 23, coloring blue squares for newspapers sold and red squares for newspapers returned.

Increase Profit

$.40 $1.20 $.80

Pages 24 and 25

Complete page 24 collectively and ask the students to do page 25 on their own.

Increase Profit

$.50 $1.00 $.50

Check page 25 collectively (answers are on page 23 at the end of this chapter).

T: Does this work help us to decide which is the best choice for a constant daily supply?

4 P

S: Shunda's profit increased until the daily supply was 26. Then when we looked at a daily supply of 27, the profit decreased (there is a sad face). So the best choice is 26.

Let the class discuss this problem briefly. The comments will likely suggest the idea indicated above, but may not be clearly stated. Some students will benefit from several peoples' explanations of the same idea.

Pages 26 and 27

Read page 26 collectively and ask the students independently to complete the balance sheet. Then check the calculations collectively.

SUPPLY	TOTAL GAIN	TOTAL LOSS	PROFIT
26	$40.80	$5.60	$35.20

Pages 28 and 29

Read and discuss these pages collectively. Lead the discussion so that the class examines page 29 and relates the break-even point (one newspaper sold balances two newspapers returned) with the determination of 26 as the best choice for a constant daily supply.

Page 30

Read this page collectively and let the students comment further if they wish about what other factors could affect Shunda's business.

Shunda's Newsstand

4:00 PM

SUPPLY

6:00 PM

On Sunday afternoon, an important football game was held; the results were in the evening paper. Shunda expected many customers. She came to her corner at 4:00 P.M. with a supply of 32 newspapers.

She sold 26 papers.

Complete Shunda's Balance Sheet and Profit-o-meter.

BALANCE SHEET		
GAIN	LOSS	PROFIT
$ 2.60	$ .30	$ 2.30

8

ON ELECTION DAY...

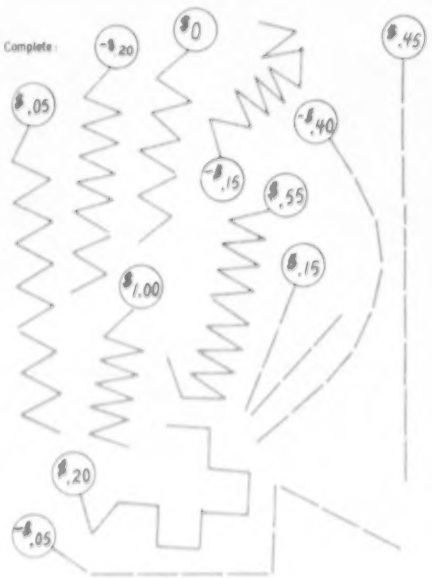
Shunda bought 36 newspapers.
She sold 30 newspapers.
She returned 6 newspapers.

SHUNDA'S Profit-o-meter

BALANCE SHEET		
GAIN	LOSS	PROFIT
$ 3.00	$ .30	$ 2.70

9

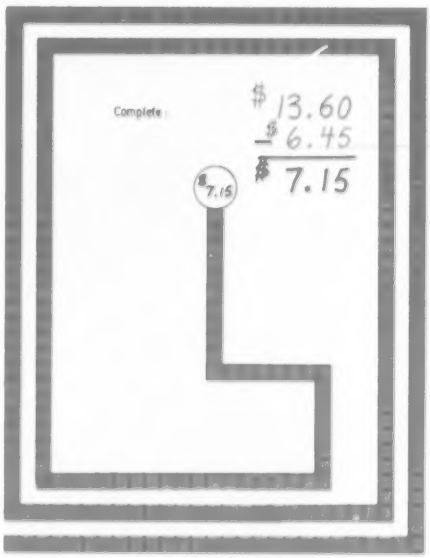
Complete:



10

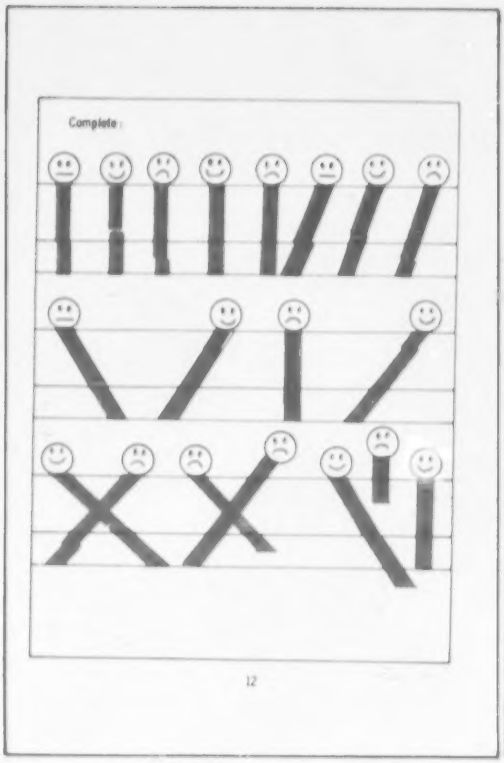
Complete:

$ 13.60
- $ 6.45
$ 7.15

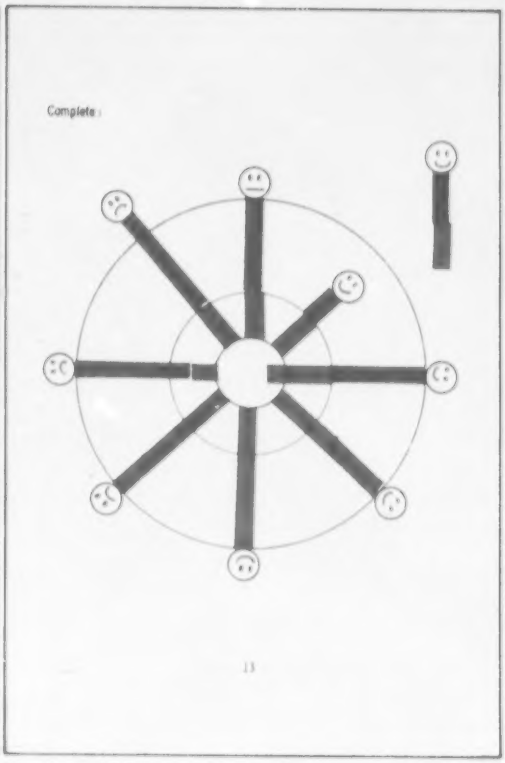


11

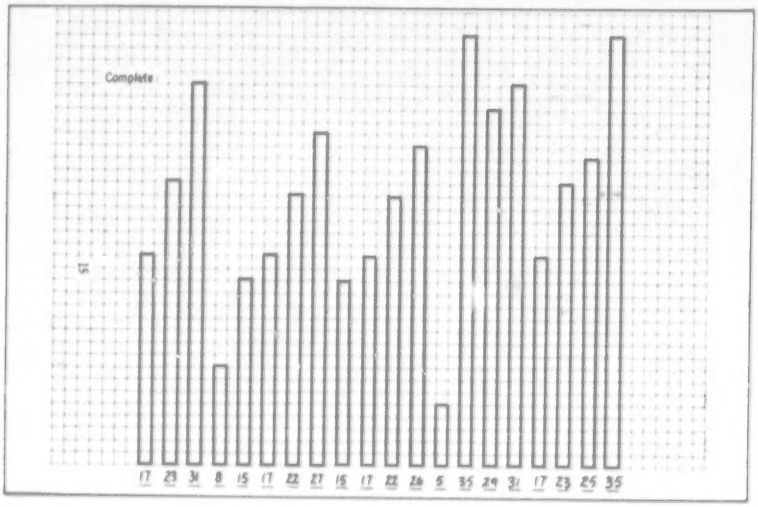
287


Complete :

12


Complete :

13


Complete
15
17 29 34 8 15 17 22 27 16 17 22 26 6 35 29 31 17 23 25 35

The largest demand during the 20 day experimental period was 35. Suppose Shunda would decide on a constant daily supply of 35 newspapers.

Page 19 shows the record (profit-o-meter) for the 20 day experimental period

IF

the constant daily supply is 35 newspapers.

Complete:

BALANCE SHEET for 20 day experimental period			
DAILY SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
35	$44.00	$13.00	$31.00

With such a supply, Shunda could satisfy her customers every day, but is it her most profitable choice?

Don't you think there are too many red squares in this record?

Suppose Shunda would decide on a constant daily supply of 20 newspapers.

Page 21 shows the record (profit-o-meter) for the 20 day experimental period

IF

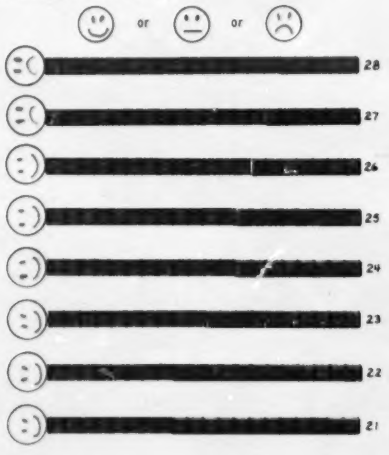
the constant daily supply is 20 newspapers.

Complete:

BALANCE SHEET for 20 day experimental period			
DAILY SUPPLY	TOTAL GAIN	TOTAL LOSS	TOTAL PROFIT
20	$35.10	$2.45	$32.65

Shunda found that 20 was a better choice than 35 for the constant daily supply. But is it the best possible choice?

Color the strips and complete the faces:



SHUNDA'S CONCLUSION

During the 20 day experimental period, the largest profit would have been obtained with a daily supply of 26 newspapers.

Complete:

BALANCE SHEET for 20 day experimental period			
SUPPLY	TOTAL GAIN	TOTAL LOSS	PROFIT
26	$40.80	$5.60	$35.20